

Distributed Estimation using Bayesian Consensus Filtering

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Abstract—We present the Bayesian consensus filter (BCF) for tracking a moving target using a networked group of sensing agents and achieving consensus on the best estimate of the probability distributions of the target's states. Our BCF framework can incorporate nonlinear target dynamic models, heterogeneous nonlinear measurement models, non-Gaussian uncertainties, and higher-order moments of the locally estimated posterior probability distribution of the target's states obtained using Bayesian filters. If the agents combine their estimated posterior probability distributions using a logarithmic opinion pool, then the sum of Kullback–Leibler divergences between the consensual probability distribution and the local posterior probability distributions is minimized. Rigorous stability and convergence results for the proposed BCF algorithm with single or multiple consensus loops are presented. Communication of probability distributions and computational methods for implementing the BCF algorithm are discussed along with a numerical example.

I. INTRODUCTION

In this paper, the term *consensus* means reaching an agreement across the network, regarding a certain subject of interest called the *target* dynamics. Distributed and networked groups of agents can sense the target, broadcast the acquired information, and reach an agreement on the gathered information using consensus algorithms. Potential applications of distributed estimation tasks include environment and pollution monitoring, tracking dust or volcanic ash clouds, tracking mobile targets such as flying objects or space debris using distributed sensor networks, etc. Consensus algorithms are extensively studied in controls [1]–[6], distributed optimization [7]–[10], and distributed estimation problems [11]–[15]. Strictly speaking, *consensus* is different from the term *distributed estimation*, which refers to finding the best estimate of the target, given a distributed network of sensing agents.

Many existing algorithms for distributed estimation [11]–[20] aim to obtain the estimated mean (first moment of the estimated probability distribution) of the target dynamics across the network, but cannot incorporate nonlinear target dynamics, heterogeneous nonlinear measurement models, non-Gaussian uncertainties, or higher-order moments of the locally estimated posterior probability distribution of the target's states. It is difficult to recursively combine local mean and covariance estimates using a linear consensus algorithm because the dimension of the vector transmitted by each agent increases linearly with time due to correlated process noise [21] and

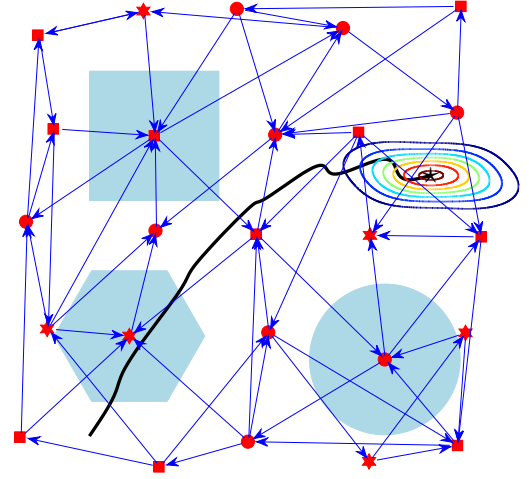


Fig. 1. A heterogeneous sensor network tracks a target, whose current position is marked by \star . The sensing region of some sensors are shown in light blue. The sensors use the Hierarchical BCF algorithm to estimate the probability distribution of the target's position and reach a consensus across the network.

the covariance update equation is usually approximated by a consensus gain [22].

Multi-agent tracking or sensing networks are deployed in a distributed fashion when the target dynamics have complex temporal and spatial variations. Hence, it is necessary to preserve the complete information captured in the locally estimated posterior probability distribution of the target's states while achieving consensus across the network. For example, while tracking an orbital debris in space, the uncertainty in position along the direction of velocity is much larger than that orthogonal to the velocity direction, and this extra information is lost if a linear consensus algorithm is used to combine the estimated mean from multiple tracking stations. As shown in Fig 1, the contour plot represents the consensual probability distribution of the target's final position, where the uncertainty along the velocity direction is relatively larger than that in the orthogonal direction.

The main objective of this paper is to extend the scope of distributed estimation algorithms to track targets with general nonlinear dynamic models with stochastic uncertainties, thereby addressing the aforementioned shortcomings. Bayesian filters [23]–[26] recursively calculate the probability density/mass function of the beliefs and update them based on new measurements. The main advantage of Bayesian filters over Kalman filter–based methods [27], [28] for estimation of nonlinear target dynamic models is that no approximation

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is needed during the filtering process. In other words, the complete information about the dynamics and uncertainties of the model can be incorporated in the filtering algorithm. However, Bayesian filtering is computationally expensive. Advances in computational capability have facilitated the implementation of Bayesian filters for robotic localization and mapping [29]–[32] as well as planning and control [33]–[35]. Practical implementation of these algorithms, in their most general form, is achieved using particle filtering [26], [36] and Bayesian programming [37], [38]. This paper focuses on developing a consensus framework for distributed Bayesian filters.

The *statistics* literature deals with the problem of reaching a consensus among individuals in a complete graph, where each individual's opinion is represented as a probability distribution [39], [40]; and under select conditions, it is shown that consensus is achieved within the group [41]–[43]. Exchange of beliefs in decentralized systems, under communication constraints, is considered in [44], [45]. Algorithms for combining probability distributions within the exponential family, i.e., a limited class of unimodal distributions that can be expressed as an exponential function, are presented in [46], [47]. If the target's states are discrete random variables, then the local estimates can be combined using a tree-search algorithm [48] or a linear consensus algorithm [49]. In contrast, this paper focuses on developing generalized Bayesian consensus algorithms with rigorous convergence analysis for achieving consensus across the network without any assumption on the shape of local prior or posterior probability distributions. The proposed distributed estimation using Bayesian consensus filtering aims to reach an agreement across the network on the best estimate, in the information theoretic sense, of the probability distribution of the target's states.

A. Paper Contributions and Organization

In this paper, we assume that agents generate their local estimate of the posterior probability distribution of the target's states using Bayesian filters with/without measurement exchange with neighbors. Then, we develop algorithms for combining these local estimates, using the logarithmic opinion pool (LogOP), to generate the consensual estimate of the probability distribution of the target's states across the network. Finally, we introduce the Bayesian consensus filter (BCF), where the local prior estimates of the target's states are first updated and the local posterior probability distributions are recursively combined during the consensus stage, so that the agents can estimate the consensual probability distribution of the target's states while simultaneously maintaining consensus across the network. The flowchart for the algorithm is shown in Fig. 2 and its pseudo-code is given in **Algorithm 1**.

The first contribution of this paper is the LogOP-based consensus algorithm for combining posterior probability distributions during the consensus stage and achieving consensus across the network. As discussed in Section III-B, this is achieved by each agent recursively communicating its posterior probability distribution of the target's states with neighboring agents and updating its estimated probability distribution

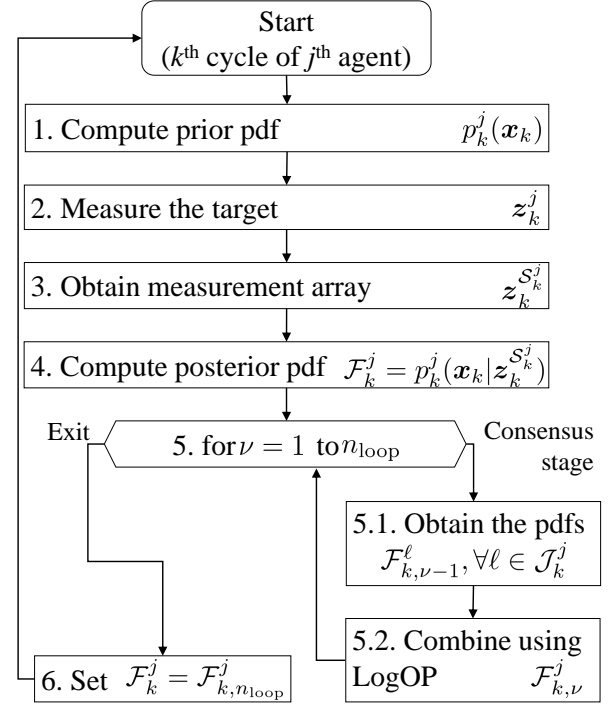


Fig. 2. Flowchart for BCF-LogOP algorithm describing the key steps for a single agent in a single time step. Steps 1–4 represent the Bayesian filtering stage, while step 5 represents the consensus stage.

of the target's states using the LogOP. As shown in Fig. 3, combining posterior probability distributions using the linear opinion pool (LinOP) typically results in multimodal solutions, which are insensitive to the weights [40]. On the other hand, combining posterior probability distributions using the LogOP typically results in unimodal, less dispersed solutions, thereby indicating a jointly preferred consensual distribution by the network. Moreover, as discussed in Section III-B, the optimal solution does not depend upon the choice of scale of the prior probability distribution and LogOP is *externally Bayesian* [40] (See Fig. 3 (c-d)).

The KL divergence is the measure of the information lost when the consensual estimate is used to approximate the locally estimated posterior probability distributions. In Theorem 6, we show that the LogOP algorithm on a strongly connected (SC) balanced graph minimizes the information lost during the consensus stage, i.e., the consensual probability distribution minimizes the sum of KL divergences with the locally estimated posterior probability distributions. Methods for communicating probability distributions and the effects of inaccuracies on the consensual probability distribution are discussed in Section III-C.

The second contribution of this paper is the BCF algorithm presented in Section IV. As illustrated in Fig. 2 and **Algorithm 1**, each agent generates a local estimate of the posterior probability distribution of the target's states using the Bayesian filter. Note that measurement exchanges with neighbors during the Bayesian filtering stage are not mandatory and can be omitted. During the consensus stage, the LogOP algorithm is executed multiple times to reach an agreement across the network. The number of consensus loops ($n_{\text{loop}} \in \mathbb{N}$) depends on the second

largest singular value of the matrix representing a SC balanced communication network topology. Moreover, the convergence conditions for a given number of consensus loops are derived. Note that this consensual probability distribution from the current time step is used as the prior probability distribution in the next time step, as shown in Fig 2. The novel features of the BCF algorithm are:

- The algorithm can be used to track targets with general nonlinear time-varying target dynamic models.
- The algorithm can be used by a SC balanced network of heterogeneous agents with general nonlinear time-varying measurement models.
- The algorithm achieves global exponential convergence across the network to the consensual probability distribution of the target's states.
- The consensual probability distribution the best estimate, in the information theoretic sense because it minimizes the sum of KL divergences with the locally estimated posterior probability distributions. If a central agent receives all the local posterior probability distributions and is tasked to find the best estimate in the information theoretic sense, then it would also yield the same consensual probability distribution. Hence, we claim to have achieved distributed estimation using the BCF algorithm.

The Hierarchical BCF algorithms, in Section IV-B, is used when some of the agents do not observe the target. In Section V, we apply the Hierarchical BCF algorithm to the problem of tracking orbital debris in space using the space surveillance network on Earth.

B. Notation

The *time index* is denoted by a right subscript. For example, \mathbf{x}_k represents the true states of the target at the k^{th} time instant. The target is always within the compact state space \mathcal{X} , i.e., $\mathbf{x}_k \in \mathcal{X}, \forall k \in \mathbb{N}$. Also, $\mathbf{x}_{1:k}$ represents an array of the true states of the target from the first to the k^{th} time instant. The *agent index* is denoted by a lower-case right superscript. For example, z_k^j represents the measurement taken by the j^{th} agent at the k^{th} time instant. The symbol $\mathbb{P}(\cdot)$ refers to probability of an event. \mathcal{F}_k^j represents the estimated probability density function (pdf) of the target's states over the state space \mathcal{X} , by the j^{th} agent at the k^{th} time instant. The symbol $p(\cdot)$ also refers to pdf or probability mass function (pmf) over the state space \mathcal{X} . During the consensus stage at the k^{th} time instant, $\mathcal{F}_{k,\nu}^j$ represents the local pdf of the target's states by the j^{th} agent at the ν^{th} consensus step and \mathcal{F}_k^* represents the consensual pdf to which each $\mathcal{F}_{k,\nu}^j$ converges. Let \mathcal{X} be the Borel σ -algebra on \mathcal{X} .

The communication network topology at the k^{th} time instant is represented by the directed time-varying graph \mathcal{G}_k , where all the agents of the system form the set of vertices \mathcal{V} (which does not change with time) and the set of directed edges is denoted by \mathcal{E}_k . The neighbors of the j^{th} agent at the k^{th} time instant is the set of agents from which the j^{th} agent receives information at the k^{th} time instant and is denoted by \mathcal{N}_k^j , i.e., $\ell \in \mathcal{N}_k^j$ if and only if $\ell_j \in \mathcal{E}_k$ for all $\ell, j \in \mathcal{V}$. The set of inclusive neighbors of the j^{th} agent is denoted by $\mathcal{J}_k^j := \mathcal{N}_k^j \cup \{j\}$.

Let \mathbb{N} and \mathbb{R} be the sets of natural numbers (positive integers) and real numbers respectively. The set of all m by n matrices over the field of real numbers \mathbb{R} is denoted by $\mathbb{R}^{m \times n}$. Let λ and σ represent the eigenvalue and the singular value of a square matrix. Let $\mathbf{1} = [1, 1, \dots, 1]^T$, \mathbf{I} , $\mathbf{0}$, and \emptyset be the ones vector, the identity matrix, the zero matrix of appropriate sizes, and the empty set respectively. The symbols $|\cdot|$, $\lceil \cdot \rceil$, and $\text{sgn}(\cdot)$ represent the absolute value, ceiling function, and signum function respectively. Let $\ln(\cdot)$ and $\log_c(\cdot)$ represent the natural logarithm and the logarithm to the base c . Finally, $\|\cdot\|_{\ell_p}$ represents the ℓ_p vector norm. The \mathcal{L}_p function denotes the set of all functions $f(\mathbf{x}) : \mathbb{R}^{n_x} \rightarrow \mathbb{R}$ with the bounded integral $(\int_{\mathcal{X}} |f(\mathbf{x})|^p d\mu(\mathbf{x}))^{1/p}$, where μ is a measure on \mathcal{X} .

II. PRELIMINARIES

In this section, we first state four assumptions used throughout this paper and then introduce the problem statement of BCF. Next, we discuss an extension of the Bayesian filter to sensor fusion over a network.

Assumption 1. In this paper, all the algorithms are presented in discrete time. \square

Assumption 2. The state space ($\mathcal{X} \subset \mathbb{R}^{n_x}$) is closed and bounded. Hence, by the Heine–Borel theorem (cf. [50, pp. 86]), \mathcal{X} is compact. \square

Assumption 3. All continuous probability distributions are upper-bounded by some large value $\mathcal{M} \in \mathbb{R}$. \square

Assumption 4. The inter-agent communication time scale is much faster than the tracking/estimation time scale. \square

Assumptions 1 and 2 are introduced to discretize the time and bound the state space, so that the algorithms are computationally tractable. Under these assumptions, particle filters [36], approximate grid-based filters, or histogram filters [32] can be used to execute the algorithms developed in this paper. Assumptions 2 and 3 are introduced to take advantage of the results in information theory and measure theory, which deal with bounded functions on compact support. Under Assumption 4, the agents can execute multiple consensus loops within each tracking time step.

We envisage that the results in this paper could be extended to continuous time if the Fokker–Plank equations are solved efficiently [51] and additional issues due to communication delay and time scale separation are addressed. Under Assumption 1, we do not discuss continuous time related issues in this paper. Next, we show that discrete and continuous probability distributions can be handled in a unified manner.

Remark 1. Let \mathcal{X} be the Borel σ -algebra for \mathcal{X} . The probability of a set $\mathcal{A} \in \mathcal{X}$ may be written as the Lebesgue–Stieltjes integral $\mathbb{P}(\mathcal{A}) = \int_{\mathcal{A}} p(\mathbf{x}) d\mu(\mathbf{x})$, where μ is a measure on \mathcal{X} . In the continuous case, $p(\mathbf{x})$ is the pdf and μ is the Lebesgue measure. In the discrete case, $p(\mathbf{x})$ is the pmf and μ is the counting measure. \square

Hence, in this paper, we only deal with pdfs over \mathcal{X} with μ as the Lebesgue measure. Similar arguments will also work for pmfs or mixed probability distributions.

A. Problem Statement

Let $\mathcal{X} \subset \mathbb{R}^{n_x}$ be the n_x -dimensional state space of the target. The dynamics of the target in discrete time $\{\mathbf{x}_k, k \in \mathbb{N}, \mathbf{x}_k \in \mathcal{X}\}$ is given by:

$$\mathbf{x}_k = \mathbf{f}_k(\mathbf{x}_{k-1}, \mathbf{v}_{k-1}), \quad (1)$$

where $\mathbf{f}_k : \mathbb{R}^{n_x} \times \mathbb{R}^{n_v} \rightarrow \mathbb{R}^{n_x}$ is a possibly nonlinear time-varying function of the state \mathbf{x}_{k-1} and an independent and identically distributed (i.i.d.) process noise \mathbf{v}_{k-1} , where n_v is the dimension of the process noise vector.

Let m heterogeneous agents simultaneously track this target and estimate the pdf of the target's states (where m does not change with time). The measurement model of the j^{th} agent is given by:

$$\mathbf{z}_k^j = \mathbf{h}_k^j(\mathbf{x}_k, \mathbf{w}_k^j), \quad \forall j \in \{1, \dots, m\}, \quad (2)$$

where $\mathbf{h}_k^j : \mathbb{R}^{n_x} \times \mathbb{R}^{n_{w_j}} \rightarrow \mathbb{R}^{n_{z_j}}$ is a possibly nonlinear time-varying function of the state \mathbf{x}_k and an i.i.d. measurement noise \mathbf{w}_k^j , where n_{z_j}, n_{w_j} are dimensions of the measurement and measurement noise vectors respectively. Note that the measurement model of agents is quite general since it accommodates heterogeneous sensors with various bandwidths, ranges, and noise characteristics and partial state observation.

The objective of the BCF is to estimate the target's states and maintain consensus across the network. This objective is achieved in two steps: (i) each agent locally estimates the pdf of the target's states using a Bayesian filter, and (ii) each agent's local estimate converges to a global estimate during the consensus stage (see Fig. 1). The objective of Bayesian filtering with/without measurement exchange, discussed in Section II-B, is to estimate the posterior pdf of the target's states at the k^{th} time instant, which is denoted by $\mathcal{F}_k^j, \forall j \in \{1, \dots, m\}$, using the estimated prior pdf of the target's states \mathcal{F}_{k-1}^j from the $(k-1)^{\text{th}}$ time instant and the new measurement array obtained at the k^{th} time instant. The objective of the consensus stage, discussed in Section III, is to guarantee pointwise convergence of each estimated pdf $\mathcal{F}_{k,\nu}^j$ to the consensual pdf \mathcal{F}_k^* .

B. Bayesian Filter with Measurement Exchange

A Bayesian filter consist of two steps: (i) the prior pdf of the target's states is obtained during the prediction stage, and (ii) the posterior pdf of the target's states is updated using the new measurement array during the update stage [23]–[26]. The Bayesian filter gives the exact posterior probability distribution, hence it is the best possible estimate of the target from the available information.

Exchange of measurements is optional since heterogeneous agents, with different priors, fields of view, resolutions, tolerances, etc., may not be able to combine measurements from other agents. For example, if a satellite in space and a low flying quadrotor are observing the same target, then they cannot exchange measurements due to their different fields of view. Furthermore, a centralized estimator may not be able to combine measurements from all heterogeneous agents in the network to estimate $p_k(\mathbf{x}_k | \mathbf{z}_k^{\{1, \dots, m\}})$, because it would have to use a common prior for all the agents. Hence, in this paper,

we let the individual agents generate their own posterior pdfs of the target's states and then combine them to get the best estimated pdf from the network.

If an agent can combine measurements from another neighboring agent during its update stage, then we call them *measurement neighbors*. In this section, we extend the Bayesian filter by assuming that each agent transmits its measurements to other agents in the network, and receives the measurements from its measurement neighbors. Let $\mathbf{z}_k^{S_k^j} := \{\mathbf{z}_k^\ell, \forall \ell \in S_k^j\}$ denote the array of measurements taken at the k^{th} time instant by the measurement neighbors of the j^{th} agent, where $S_k^j \subseteq \mathcal{J}_k^j$ denotes the set of measurement neighbors among the inclusive neighbors of the j^{th} agent. Next, we assume that the prior is available at the initial time.

Assumption 5. For each agent j , the initial prior of the states $\mathcal{F}_0^j = p_0^j(\mathbf{x}_0)$, is assumed to be available. In case no knowledge about \mathbf{x}_0 is available, \mathcal{F}_0^j is assumed to be uniformly distributed over \mathcal{X} . \square

In *Bayesian Filtering with Measurement Exchanges*, the j^{th} agent estimates the posterior pdf of the target's states $\mathcal{F}_k^j = p_k^j(\mathbf{x}_k | \mathbf{z}_k^{S_k^j})$ at the k^{th} time instant using the estimated consensual pdf of the target's states $\mathcal{F}_{k-1}^j = p_{k-1}^j(\mathbf{x}_{k-1})$ from the $(k-1)^{\text{th}}$ time instant and the new measurement array $\mathbf{z}_k^{S_k^j}$ obtained at the k^{th} time instant. The prediction stage involves using the target dynamics model (1) to obtain the estimated pdf of the target's states at the k^{th} time instant via the Chapman–Kolmogorov equation:

$$p_k^j(\mathbf{x}_k) = \int_{\mathcal{X}} p_k^j(\mathbf{x}_k | \mathbf{x}_{k-1}) p_{k-1}^j(\mathbf{x}_{k-1}) d\mu(\mathbf{x}_{k-1}) \quad (3)$$

The probabilistic model of the state evolution $p_k^j(\mathbf{x}_k | \mathbf{x}_{k-1})$ is defined by the target dynamics model (1) and the known statistics of the i.i.d. process noise \mathbf{v}_{k-1} .

Proposition 1. The new measurement array $(\mathbf{z}_k^{S_k^j})$ is used to compute the posterior pdf of the target's states $(\mathcal{F}_k^j = p_k^j(\mathbf{x}_k | \mathbf{z}_k^{S_k^j}))$ during the update stage using Bayes' rule (4):

$$p_k^j(\mathbf{x}_k | \mathbf{z}_k^{S_k^j}) = \frac{\left(\prod_{\ell \in S_k^j} p_k^\ell(\mathbf{z}_k^\ell | \mathbf{x}_k) \right) p_k^j(\mathbf{x}_k)}{\int_{\mathcal{X}} \left(\prod_{\ell \in S_k^j} p_k^\ell(\mathbf{z}_k^\ell | \mathbf{x}_k) \right) p_k^j(\mathbf{x}_k) d\mu(\mathbf{x}_k)}, \quad (4)$$

The likelihood function $p_k^\ell(\mathbf{z}_k^\ell | \mathbf{x}_k), \forall \ell \in S_k^j$ is defined by the measurement model (2), and the corresponding known statistics of the i.i.d. measurement noise \mathbf{w}_k^ℓ .

Proof: We need to show that the term $p(\mathbf{z}_k^{S_k^j} | \mathbf{x}_k)$ in the Bayesian filter [26] simplifies to $\left(\prod_{\ell \in S_k^j} p_k^\ell(\mathbf{z}_k^\ell | \mathbf{x}_k) \right)$. Let the agents $r, r+1, \dots$ be measurement neighbors of the j^{th} agent at the k^{th} time instant (i.e., $r, r+1, \dots \in \mathcal{N}_k^j$). Let us define $\mathbf{z}_k^{S_k^j \setminus \{r\}} := \{\mathbf{z}_k^\ell, \forall \ell \in S_k^j \setminus \{r\}\}$ as the measurement array obtained by the j^{th} agent at the k^{th} time instant, which does not contain the measurement from the r^{th} agent. Since the measurement noise is i.i.d., (2) describes a Markov process of order one, we get:

$$\begin{aligned} \frac{p(z_k^{S_k^j}, \mathbf{x}_k)}{p(\mathbf{x}_k)} &= \frac{p(z_k^{S_k^j}, \mathbf{x}_k)}{p(z_k^{S_k^j \setminus \{r\}}, \mathbf{x}_k)} \frac{p(z_k^{S_k^j \setminus \{r\}}, \mathbf{x}_k)}{p(z_k^{S_k^j \setminus \{r, r+1\}}, \mathbf{x}_k)} \dots \frac{p(z_k^j, \mathbf{x}_k)}{p(\mathbf{x}_k)} \\ &= p(z_k^r | z_k^{S_k^j \setminus \{r\}}, \mathbf{x}_k) p(z_k^{(r+1)} | z_k^{S_k^j \setminus \{r, r+1\}}, \mathbf{x}_k) \dots p(z_k^j | \mathbf{x}_k) \end{aligned}$$

Thus, we obtain $p(z_k^{S_k^j} | \mathbf{x}_k) = p(z_k^{S_k^j}, \mathbf{x}_k) / p(\mathbf{x}_k) = \prod_{\ell \in S_k^j} p_k^\ell(z_k^\ell | \mathbf{x}_k)$. ■

If the estimates are represented by pmfs, then we can compare these estimates using entropy [52, pp. 13], which is a measure of the uncertainty associated with a random variable or its information content.

Remark 2. Let \mathbf{Y}_k^j be a random variable with the pmf $p_k^j(\mathbf{x}_k | z_k^j)$ given by a stand-alone Bayesian filter [26], $\mathbf{Y}_k^{S_k^j}$ is a random variable with the pmf $p_k^j(\mathbf{x}_k | z_k^{S_k^j})$ given by Bayesian filter with measurement exchange, and $H(\cdot)$ refers to the entropy of the random variable. Since \mathbf{Y}_k^j is obtained by conditioning \mathbf{Y}_k^j with $z_k^{S_k^j \setminus \{j\}}$ because $p_k^j(\mathbf{x}_k | z_k^{S_k^j}) = p_k^j(\mathbf{x}_k | z_k^j, z_k^{S_k^j \setminus \{j\}})$, the claim follows from the theorem on conditioning reduces entropy (cf. [52, pp. 27]):

$$0 \leq I(\mathbf{Y}_k^j, z_k^{S_k^j \setminus \{j\}}) = H(\mathbf{Y}_k^j) - H(\mathbf{Y}_k^j | z_k^{S_k^j \setminus \{j\}}),$$

where $I(\cdot; \cdot)$ refers to the nonnegative mutual information between two random variables. Since $H(\mathbf{Y}_k^{S_k^j}) = H(\mathbf{Y}_k^j | z_k^{S_k^j \setminus \{j\}}) \leq H(\mathbf{Y}_k^j)$, the estimate given by Bayesian filter with measurement exchange is more accurate than that obtained by a stand-alone Bayesian filter. □

Note that (4) is similar to the empirical equation for *Independent Likelihood Pool* given in [35] and a generalization of the *Distributed Sequential Bayesian Estimation Algorithm* given in [53]. The structure of (4) ensures that an arbitrary part of the prior distribution does not dominate the measurements. There is no consensus protocol across the network because each agent receives information only from its neighboring agents and never receives measurements (even indirectly) from any other agent in the network.

III. COMBINING PROBABILITY DISTRIBUTIONS

In this section, we present the algorithms for achieving consensus in probability distributions across the network. As discussed before, the objective of the consensus stage in **Algorithm 1** is to guarantee pointwise convergence of each \mathcal{F}_k^j to a consensual pdf \mathcal{F}_k^* , which is independent of j . This is achieved by each agent recursively transmitting its estimated pdf of the target's states to other agents, receiving estimates of its neighboring agents, and updating its estimate of the target. Let $\mathcal{F}_{k,0}^j = \mathcal{F}_k^j$ represent the local estimated posterior pdf of the target's states, by the j^{th} agent at the start of the consensus stage, obtained using Bayesian filters with/without measurement exchange. During each of the n_{loop} iterations within the consensus stage in **Algorithm 1**, this estimate is updated as follows:

$$\mathcal{F}_{k,\nu}^j = \mathcal{T} \left(\bigcup_{\ell \in \mathcal{J}_k^j} \{\mathcal{F}_{k,\nu-1}^\ell\} \right), \forall j \in \{1, \dots, m\}, \forall \nu \in \mathbb{N}, \quad (5)$$

where $\mathcal{T}(\cdot)$ is the linear or logarithmic opinion pool for combining the pdf estimates. Note that the problem of measurement neighbors does not arise here since all pdfs are expressed over the complete state space \mathcal{X} .

We introduce Lemma 2 to show that pointwise convergence of pdfs is the sufficient condition for convergence of their induced measures in total variation (TV). Let $\mathcal{F}_1, \dots, \lim_{n \rightarrow \infty} \mathcal{F}_n, \mathcal{F}^*$ be real-valued measurable functions on \mathcal{X} , \mathcal{X} be the Borel σ -algebra of \mathcal{X} , and \mathcal{A} be any set in \mathcal{X} . If $\mu_{\mathcal{F}_n}(\mathcal{A}) = \int_{\mathcal{A}} \mathcal{F}_n(\mathbf{x}) d\mu(\mathbf{x})$ for any set $\mathcal{A} \in \mathcal{X}$, then $\mu_{\mathcal{F}_n}$ is defined as the measure induced by the function \mathcal{F}_n on \mathcal{X} . Let $\mu_{\mathcal{F}_n}, \mu_{\mathcal{F}^*}$ denote the respective induced measures of $\mathcal{F}_n, \mathcal{F}^*$ on \mathcal{X} .

Definition 3. (Convergence in TV) If $\|\mu_{\mathcal{F}_n} - \mu_{\mathcal{F}^*}\|_{\text{TV}} := \sup_{\mathcal{A} \in \mathcal{X}} |\mu_{\mathcal{F}_n}(\mathcal{A}) - \mu_{\mathcal{F}^*}(\mathcal{A})|$ tends to zero as $n \rightarrow \infty$, then the measure $\mu_{\mathcal{F}_n}$ converges to the measure $\mu_{\mathcal{F}^*}$ in TV, i.e., $\lim_{n \rightarrow \infty} \mu_{\mathcal{F}_n} \xrightarrow{\text{T.V.}} \mu_{\mathcal{F}^*}$. □

Lemma 2. (Pointwise convergence implies convergence in TV) If \mathcal{F}_n converges to \mathcal{F}^* pointwise, i.e., $\lim_{n \rightarrow \infty} \mathcal{F}_n = \mathcal{F}^*$ pointwise; then the measure $\mu_{\mathcal{F}_n}$ converges in TV to the measure $\mu_{\mathcal{F}^*}$, i.e., $\lim_{n \rightarrow \infty} \mu_{\mathcal{F}_n} \xrightarrow{\text{T.V.}} \mu_{\mathcal{F}^*}$.

Proof: Similar to the proof of Scheffé's theorem [54, pp. 84], under Assumption 3, using the dominated convergence theorem (cf. [54, Theorem 1.5.6, pp. 23]) for any set $\mathcal{A} \in \mathcal{X}$ gives:

$$\lim_{n \rightarrow \infty} \int_{\mathcal{A}} \mathcal{F}_n(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\mathcal{A}} \lim_{n \rightarrow \infty} \mathcal{F}_n(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\mathcal{A}} \mathcal{F}^*(\mathbf{x}) d\mu(\mathbf{x}).$$

This relation between measures implies that $\|\lim_{n \rightarrow \infty} \mu_{\mathcal{F}_n} - \mu_{\mathcal{F}^*}\|_{\text{TV}} = 0$ and $\lim_{n \rightarrow \infty} \mu_{\mathcal{F}_n} \xrightarrow{\text{T.V.}} \mu_{\mathcal{F}^*}$. ■

A. Consensus using the Linear Opinion Pool

The first method of combining the estimates is motivated by the linear consensus algorithms widely studied in the literature [5]–[7]. The pdfs are combined using the Linear Opinion Pool (LinOP) of probability measures [39], [40]:

$$\mathcal{F}_{k,\nu}^j = \sum_{\ell \in \mathcal{J}_k^j} a_{k,\nu-1}^{j\ell} \mathcal{F}_{k,\nu-1}^\ell, \forall j \in \{1, \dots, m\}, \forall \nu \in \mathbb{N}, \quad (6)$$

where $\sum_{\ell \in \mathcal{J}_k^j} a_{k,\nu-1}^{j\ell} = 1$ and the updated pdf $\mathcal{F}_{k,\nu}^j$ after the ν^{th} consensus loop is a weighted average of the pdfs of the inclusive neighbors $\mathcal{F}_{k,\nu-1}^\ell, \forall \ell \in \mathcal{J}_k^j$ from the $(\nu-1)^{\text{th}}$ consensus loop, at the k^{th} time instant. Let $\mathcal{W}_{k,\nu} := (\mathcal{F}_{k,\nu-1}^1, \dots, \mathcal{F}_{k,\nu-1}^m)^T$ denote an array of pdf estimates of all the agents after the ν^{th} consensus loop, then the LinOP (6) can be expressed concisely as:

$$\mathcal{W}_{k,\nu} = P_{k,\nu-1} \mathcal{W}_{k,\nu-1}, \forall \nu \in \mathbb{N}, \quad (7)$$

where $P_{k,\nu-1}$ is a matrix with entries $P_{k,\nu-1}[j, \ell] = a_{k,\nu-1}^{j\ell}$.

Assumption 6. The communication network topology of the multi-agent system \mathcal{G}_k is strongly connected (SC). The weights $a_{k,\nu-1}^{j\ell}, \forall j, \ell \in \{1, \dots, m\}$ and the matrix $P_{k,\nu-1}$ have the following properties: (i) the weights are the same for all consensus loops within each time instant, i.e., $a_{k,\nu-1}^{j\ell} = a_k^{j\ell}$.

and $P_{k,\nu-1} = P_k, \forall \nu \in \mathbb{N}$; (ii) the matrix P_k conforms with the graph \mathcal{G}_k , i.e., $a_k^{j\ell} > 0$ if and only if $\ell \in \mathcal{J}_k^j$, else $a_k^{j\ell} = 0$; and (iii) the matrix P_k is row stochastic, i.e., $\sum_{\ell=1}^m a_k^{j\ell} = 1$. \square

Theorem 3. (Consensus using the LinOP on SC Digraphs) *Under Assumption 6, using the LinOP (6), each $\mathcal{F}_{k,\nu}^j$ asymptotically converges pointwise to the pdf $\mathcal{F}_k^* = \sum_{i=1}^m \pi_i \mathcal{F}_{k,0}^i$ where $\pi = [\pi_1, \dots, \pi_m]^T$ is the unique stationary distribution of P_k . Furthermore, their induced measures converge in total variation, i.e., $\lim_{\nu \rightarrow \infty} \mu_{\mathcal{F}_{k,\nu}^j} \xrightarrow{TV} \mu_{\mathcal{F}_k^*}, \forall j \in \{1, \dots, m\}$.*

Proof: See Appendix A. \blacksquare

Theorem 3 is a generalization of the linear consensus algorithm for combining joint measurement probabilities [55]. Moreover, if $\pi = \mathbf{1}$ and each $\mathcal{F}_{k,0}^j$ is a \mathcal{L}_2 function, then $\mathcal{F}_k^* = \frac{1}{m} \sum_{i=1}^m \mathcal{F}_{k,0}^i$ globally minimizes the sum of the squares of \mathcal{L}_2 distances with the locally estimated posterior pdfs.

As shown in Fig. 3 (a-b), the main difficulty with the LinOP is that the resulting solution is typically multimodal, so no clear choice for jointly preferred estimate emerges from it [40]. Moreover, the LinOP algorithm critically depends on the assumption that the same 0-1 scale is used by every agent as shown in Fig. 3 (c-d). Hence, better schemes for combining probability distributions are needed for the proposed BCF algorithm.

B. Consensus using the Logarithmic Opinion Pool

Note that $\mathcal{F}_{k,\nu}^j = p_{k,\nu}^j(\mathbf{x}_k), \forall \mathbf{x}_k \in \mathcal{X}$ represents the pdf of the estimated target's states by the j^{th} agent during the ν^{th} consensus loop at the k^{th} time instant. The LogOP is given as [41]:

$$\mathcal{F}_{k,\nu}^j = p_{k,\nu}^j(\mathbf{x}_k) = \frac{\prod_{\ell \in \mathcal{J}_k^j} \left(p_{k,\nu-1}^\ell(\mathbf{x}_k) \right)^{a_{k,\nu-1}^{j\ell}}}{\int_{\mathcal{X}} \prod_{\ell \in \mathcal{J}_k^j} \left(p_{k,\nu-1}^\ell(\mathbf{x}_k) \right)^{a_{k,\nu-1}^{j\ell}} d\mu(\mathbf{x}_k)}, \quad \forall j \in \{1, \dots, m\}, \forall \nu \in \mathbb{N}, \quad (8)$$

where $\sum_{\ell \in \mathcal{J}_k^j} a_{k,\nu-1}^{j\ell} = 1$ and the integral in the denominator of (8) is finite. Thus the updated pdf $\mathcal{F}_{k,\nu}^j$ after the ν^{th} consensus loop is the weighted geometric average of the pdfs of the inclusive neighbors $\mathcal{F}_{k,\nu-1}^\ell, \forall \ell \in \mathcal{J}_k^j$ from the $(\nu-1)^{\text{th}}$ consensus loop, at the k^{th} time instant. As shown in Fig. 3 (a-b), the LogOP solution is typically unimodal and less dispersed, indicating a consensual estimate jointly preferred by the network [40]. As shown in Fig. 3 (c-d), the LogOP solution is invariant under rescaling of individual degrees of belief, hence it preserves an important credo of uni-Bayesian decision theory; i.e., the optimal decision should not depend upon the choice of scale for the utility function or prior probability distribution [56]. When the parameter space is finite and a 0-1 probability scale is adopted, the LogOP is equivalent to the Nash product [57]. Note that if the local probability distribution of the target's states is inherently multimodal, as shown in Fig. 3 (e-f), then LogOP preserves this multimodal nature while combining these local estimates. The most compelling reason for using LogOP is that it is

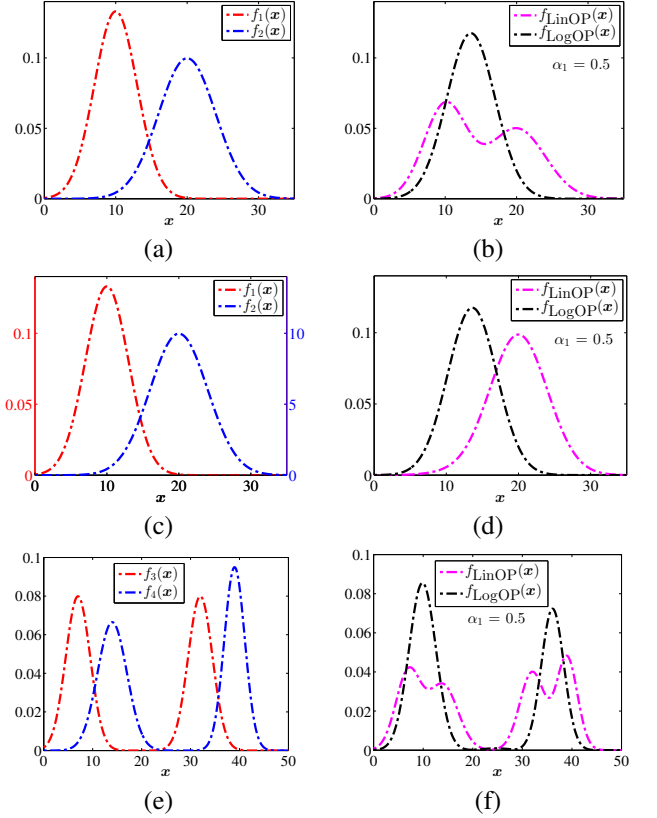


Fig. 3. In (a), two unimodal pdfs $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ are shown. In (b), these pdfs are combined using the LinOP and LogOP using the weight $\alpha_1 = 0.5$, i.e., $f_{\text{LinOP}}(\mathbf{x}) = (\alpha_1 f_1(\mathbf{x}) + (1 - \alpha_1) f_2(\mathbf{x}))$ and $f_{\text{LogOP}}(\mathbf{x}) = (f_1^{\alpha_1} \times f_2^{(1-\alpha_1)}) / \left(\int_{\mathcal{X}} (f_1^{\alpha_1} \times f_2^{(1-\alpha_1)}) d\mu(\mathbf{x}) \right)$. Note that the LinOP solution is multimodal while the LogOP solution is unimodal, indicating a consensual pdf. In (c), the scale of the function $f_2(\mathbf{x})$ is changed to 0-100 from the standard 0-1 scale. In (d), the normalized LinOP solution changes drastically but the LogOP solution remains unaffected. In (e), the pdfs $f_3(\mathbf{x})$ and $f_4(\mathbf{x})$ have bimodal nature. In (f), the LogOP solution preserves this bimodal nature.

externally Bayesian; i.e., finding the consensus distribution commutes with the process of revising distributions using a commonly agreed likelihood distribution. Thus [40]:

$$\begin{aligned} & \mathcal{T} \left(\bigcup_{\ell \in \mathcal{J}_k^j} \left\{ \frac{l(\mathbf{x}) p_k^\ell(\mathbf{x})}{\int_{\mathcal{X}} l(\mathbf{x}) p_k^\ell(\mathbf{x}) d\mu(\mathbf{x})} \right\} \right) \\ &= \frac{l(\mathbf{x}) \mathcal{T} \left(\bigcup_{\ell \in \mathcal{J}_k^j} \{ p_k^\ell(\mathbf{x}) \} \right)}{\int_{\mathcal{X}} l(\mathbf{x}) \mathcal{T} \left(\bigcup_{\ell \in \mathcal{J}_k^j} \{ p_k^\ell(\mathbf{x}) \} \right) d\mu(\mathbf{x})}, \end{aligned} \quad (9)$$

where $\mathcal{T}(\cdot)$ refers to the LogOP (8), $p_k^\ell(\mathbf{x}), \forall \ell \in \mathcal{J}_k^j$ are pdfs on \mathcal{X} and $l(\mathbf{x})$ is an arbitrary likelihood pdf on \mathcal{X} . Due to these advantages, LogOP is used for combining prior distributions [58] and conditional random fields for natural language processing tasks [59]. Next, we present consensus theorems using the LogOP.

Assumption 7. The local estimated pdf at the start of the consensus stage is positive everywhere, i.e., $\mathcal{F}_{k,0}^j = p_{k,0}^j(\mathbf{x}_k) > 0, \forall \mathbf{x}_k \in \mathcal{X}, \forall j \in \{1, \dots, m\}$. \square

Assumption 7 is introduced to avoid regions with zero probability, since they would constitute vetoes and unduly great emphasis would get placed on them. Moreover, the LogOP

guarantees that $\mathcal{F}_{k,\nu}^j$ will remain positive for all subsequent consensus loop.

Definition 4. ($\mathcal{H}_{k,\nu}^j$ vector for LogOP) For the purpose of analysis, let us choose $\psi_k \in \mathcal{X}$ such that $p_{k,\nu}^j(\psi_k) > 0, \forall j \in \{1, \dots, m\}, \forall \nu \in \mathbb{N}$. Let us define $\mathcal{H}_{k,\nu}^j := \ln \left[\frac{p_{k,\nu}^j(\mathbf{x}_k)}{p_{k,\nu}^j(\psi_k)} \right]$. Under Assumption 7, $\mathcal{H}_{k,\nu}^j$ is a well-defined function, but need not be a \mathcal{L}_1 function. Then, by simple algebraic manipulation of (8), we get [60]:

$$\frac{p_{k,\nu}^j(\mathbf{x}_k)}{p_{k,\nu}^j(\psi_k)} = \frac{\left(\frac{\prod_{\ell \in \mathcal{J}_k^j} (p_{k,\nu-1}^\ell(\mathbf{x}_k))^{a_{k,\nu-1}^{j\ell}}}{\int_{\mathcal{X}} \prod_{\ell \in \mathcal{J}_k^j} (p_{k,\nu-1}^\ell(\mathbf{x}_k))^{a_{k,\nu-1}^{j\ell}} d\mu(\mathbf{x}_k)} \right)}{\left(\frac{\prod_{\ell \in \mathcal{J}_k^j} (p_{k,\nu-1}^\ell(\psi_k))^{a_{k,\nu-1}^{j\ell}}}{\int_{\mathcal{X}} \prod_{\ell \in \mathcal{J}_k^j} (p_{k,\nu-1}^\ell(\mathbf{x}_k))^{a_{k,\nu-1}^{j\ell}} d\mu(\mathbf{x}_k)} \right)},$$

$$\mathcal{H}_{k,\nu}^j = \sum_{\ell \in \mathcal{J}_k^j} a_{k,\nu-1}^{j\ell} \mathcal{H}_{k,\nu-1}^\ell, \forall j \in \{1, \dots, m\}, \nu \in \mathbb{N}. \quad (10)$$

Note that (10) is similar to the LinOP (6). Let $\mathcal{U}_{k,\nu} := \left(\mathcal{H}_{k,\nu}^1, \dots, \mathcal{H}_{k,\nu}^m \right)^T$ be an array of the estimates of all the agents during the ν^{th} consensus loop at the k^{th} time instant, then the equation (10) can be expressed concisely as:

$$\mathcal{U}_{k,\nu} = P_{k,\nu-1} \mathcal{U}_{k,\nu-1}, \forall \nu \in \mathbb{N}, \quad (11)$$

where $P_{k,\nu-1}$ is a matrix with entries $a_{k,\nu-1}^{jl}$. \square

Thus we are able to use the highly nonlinear LogOP for combining the pdf estimates, but we have reduced the complexity of the problem to that of consensus using the LinOP.

Theorem 4. (Consensus using the LogOP on SC Digraphs) Under Assumptions 6 and 7, using the LogOP (8), each $\mathcal{F}_{k,\nu}^j$ asymptotically converges pointwise to the pdf \mathcal{F}_k^* given by:

$$\mathcal{F}_k^* = p_k^*(\mathbf{x}_k) = \frac{\prod_{i=1}^m \left(p_{k,0}^i(\mathbf{x}_k) \right)^{\pi_i}}{\int_{\mathcal{X}} \prod_{i=1}^m \left(p_{k,0}^i(\mathbf{x}_k) \right)^{\pi_i} d\mu(\mathbf{x}_k)}, \quad (12)$$

where π is the unique stationary distribution of P_k . Furthermore, their induced measures converge in total variation, i.e., $\lim_{\nu \rightarrow \infty} \mu_{\mathcal{F}_{k,\nu}^j} \xrightarrow{\text{T.V.}} \mu_{\mathcal{F}_k^*}, \forall j \in \{1, \dots, m\}$.

Proof: Similar to the proof of Theorem 3, each $\mathcal{H}_{k,\nu}^j$ converges pointwise to $\mathcal{H}_k^* = \pi^T \mathcal{U}_{k,0} = \sum_{i=1}^m \pi_i \mathcal{H}_{k,0}^i$ asymptotically. We additionally need to show that convergence of $\mathcal{H}_{k,\nu}^j$ to \mathcal{H}_k^* implies pointwise convergence of $\mathcal{F}_{k,\nu}^j$ to \mathcal{F}_k^* . We have $\forall \mathbf{x}_k \in \mathcal{X}$:

$$\lim_{\nu \rightarrow \infty} \left(\ln p_{k,\nu}^j(\mathbf{x}_k) - \ln p_{k,\nu}^j(\psi_k) \right) = \ln p_k^*(\mathbf{x}_k) - \ln p_k^*(\psi_k). \quad (13)$$

We claim $\exists \bar{\psi}_k \in \mathcal{X}$ such that $\lim_{\nu \rightarrow \infty} p_{k,\nu}^j(\bar{\psi}_k) = p_k^*(\bar{\psi}_k)$. If this claim is untrue, then $0 < \lim_{\nu \rightarrow \infty} p_{k,\nu}^j(\mathbf{x}_k) < p_k^*(\mathbf{x}_k), \forall \mathbf{x}_k \in \mathcal{X}$ or vice versa. Hence $\int_{\mathcal{X}} \lim_{\nu \rightarrow \infty} p_{k,\nu}^j(\mathbf{x}_k) d\mu(\mathbf{x}_k) = 1 < \int_{\mathcal{X}} p_k^*(\mathbf{x}_k) d\mu(\mathbf{x}_k)$, which results in contradiction since $p_k^*(\mathbf{x}_k)$ is also a pdf. Hence, substituting $\bar{\psi}_k$ into equation (13) gives $\lim_{\nu \rightarrow \infty} p_{k,\nu}^j(\mathbf{x}_k) = p_k^*(\mathbf{x}_k), \forall \mathbf{x}_k \in \mathcal{X}$. Thus each $\mathcal{F}_{k,\nu}^j$

converges pointwise to the consensual pdf \mathcal{F}_k^* given by (12). By Lemma 2, the measure induced by $\mathcal{F}_{k,\nu}^j$ on \mathcal{X} converges in total variation to the measure induced by \mathcal{F}_k^* on \mathcal{X} , i.e., $\lim_{\nu \rightarrow \infty} \mu_{\mathcal{F}_{k,\nu}^j} \xrightarrow{\text{T.V.}} \mu_{\mathcal{F}_k^*}$. \blacksquare

Since Perron–Frobenius theorem only yields asymptotic convergence, we next discuss the algorithm for achieving global exponential convergence using balanced graphs.

Assumption 8. In addition to Assumption 6, the weights $a_{k,\nu}^{j\ell}$ are such that the digraph \mathcal{G}_k is balanced. Hence for every vertex, the in-degree equals the out-degree, i.e., $\sum_{\ell \in \mathcal{J}_k^j} a_{k,\nu}^{j\ell} = \sum_{r|j \in \mathcal{J}_k^r} a_{k,\nu}^{rj}$, where $j, \ell, r \in \{1, \dots, m\}$. \square

Theorem 5. (Consensus using the LogOP on SC Balanced Digraphs) Under Assumption 7 and 8, using the LogOP (8), each $\mathcal{F}_{k,\nu}^j$ globally exponentially converges pointwise to the pdf \mathcal{F}_k^* given by:

$$\mathcal{F}_k^* = p_k^*(\mathbf{x}_k) = \frac{\prod_{i=1}^m \left(p_{k,0}^i(\mathbf{x}_k) \right)^{\frac{1}{m}}}{\int_{\mathcal{X}} \prod_{i=1}^m \left(p_{k,0}^i(\mathbf{x}_k) \right)^{\frac{1}{m}} d\mu(\mathbf{x}_k)} \quad (14)$$

at a rate faster or equal to $\sqrt{\lambda_{m-1}(P_k^T P_k)} = \sigma_{m-1}(P_k)$. Furthermore, their induced measures globally exponentially converge in total variation, i.e., $\lim_{\nu \rightarrow \infty} \mu_{\mathcal{F}_{k,\nu}^j} \xrightarrow{\text{T.V.}} \mu_{\mathcal{F}_k^*}, \forall j \in \{1, \dots, m\}$.

Proof: Since Assumption 8 is stronger than Assumption 6, we get $\lim_{\nu \rightarrow \infty} P_k^\nu = \mathbf{1}\pi^T$. Moreover, since P_k is also a column stochastic matrix, therefore $\pi = \frac{1}{m}\mathbf{1}$ is its left eigenvector corresponding to the eigenvalue 1, i.e., $P_k^T \frac{1}{m}\mathbf{1} = \frac{1}{m}\mathbf{1}$ and satisfying the normalizing condition. Hence, we get $\lim_{\nu \rightarrow \infty} P_k^\nu = \frac{1}{m}\mathbf{1}\mathbf{1}^T$ and each $\mathcal{H}_{k,\nu}^j$ converges pointwise to $\mathcal{H}_k^* = \frac{1}{m}\mathbf{1}^T \mathcal{U}_{k,0} = \frac{1}{m} \sum_{i=1}^m \mathcal{H}_{k,0}^i$.

From the proof of Theorem 4, we get that each $\mathcal{F}_{k,\nu}^j$ converges pointwise to the consensual pdf \mathcal{F}_k^* given by (14). Note that $\mathcal{F}_{k,\nu}^j$ are \mathcal{L}_1 functions but $\mathcal{H}_{k,\nu}^j$ need not be \mathcal{L}_1 functions. Let $V_{\text{tr}} = \left[\frac{1}{\sqrt{m}}\mathbf{1}, V_s \right]$ be the orthonormal matrix of eigenvectors of the symmetric primitive matrix $P_k^T P_k$. By spectral decomposition [61], we get:

$$V_{\text{tr}}^T P_k^T P_k V_{\text{tr}} = \begin{bmatrix} 1 & \mathbf{0}^{1 \times (m-1)} \\ \mathbf{0}^{(m-1) \times 1} & V_s^T P_k^T P_k V_s \end{bmatrix},$$

where $\frac{1}{m}\mathbf{1}^T P_k^T P_k \mathbf{1} = 1$, $\frac{1}{\sqrt{m}}\mathbf{1}^T P_k^T P_k V_s = \mathbf{0}^{1 \times (m-1)}$, and $V_s^T P_k^T P_k \mathbf{1} \frac{1}{\sqrt{m}} = \mathbf{0}^{(m-1) \times 1}$ are used. Since the eigenvectors are orthonormal, $V_s V_s^T + \frac{1}{m}\mathbf{1}\mathbf{1}^T = \mathbf{I}$. The rate at which $\mathcal{U}_{k,\nu}$ synchronizes to $\frac{1}{\sqrt{m}}\mathbf{1}$ (or \mathcal{U}_k^*) is equal to the rate at which $V_s^T \mathcal{U}_{k,\nu} \rightarrow \mathbf{0}^{(m-1) \times 1}$. Pre-multiplying (11) by V_s^T and substituting $V_s^T \mathbf{1} = 0$ results in:

$$V_s^T \mathcal{U}_{k,\nu} = V_s^T P_k \left(V_s V_s^T + \frac{1}{m}\mathbf{1}\mathbf{1}^T \right) \mathcal{U}_{k,\nu-1} \\ = V_s^T P_k V_s V_s^T \mathcal{U}_{k,\nu-1}.$$

Let $\mathbf{z}_{k,\nu} = V_s^T \mathcal{U}_{k,\nu}$. The corresponding virtual dynamics is represented by $\mathbf{z}_{k,\nu} = (V_s^T P_k V_s) \mathbf{z}_{k,\nu-1}$, which has both $V_s^T \mathcal{U}_{k,\nu}$ and $\mathbf{0}$ as particular solutions. Let $\Phi_{k,\nu} = \mathbf{z}_{k,\nu}^T \mathbf{z}_{k,\nu}$ be a candidate Lyapunov function for this dynamics. Expanding this gives:

$$\Phi_{k,\nu} = \mathbf{z}_{k,\nu-1}^T V_s^T P_k^T P_k V_s \mathbf{z}_{k,\nu-1} \leq (\lambda_{\max}(V_s^T P_k^T P_k V_s)) \Phi_{k,\nu-1}.$$

Note that $V_s^T P_k^T P_k V_s$ contains all the eigenvalues of $P_k^T P_k$ other than 1. Hence $\lambda_{\max}(V_s^T P_k^T P_k V_s) = \lambda_{m-1}(P_k^T P_k) < 1$ and $\Phi_{k,\nu}$ globally exponentially vanishes with a rate faster or equal to $\lambda_{m-1}(P_k^T P_k)$. Hence each $\mathcal{H}_{k,\nu}^j$ globally exponentially converges pointwise to \mathcal{H}_k^* with a rate faster or equal to $\sqrt{\lambda_{m-1}(P_k^T P_k)} = \sigma_{m-1}(P_k)$.

Next, we need to find the rate of convergence of $\mathcal{F}_{k,\nu}^j$ to \mathcal{F}_k^* . From the exponential convergence of $\mathcal{H}_{k,\nu}^j$, we get:

$$\left| \ln \left[\frac{p_{k,\nu}^j(\mathbf{x}_k)}{p_k^*(\mathbf{x}_k)} \frac{p_k^*(\psi_k)}{p_{k,\nu}^j(\psi_k)} \right] \right| \leq \sigma_{m-1}(P_k) \left| \ln \left[\frac{p_{k,\nu-1}^j(\mathbf{x}_k)}{p_k^*(\mathbf{x}_k)} \frac{p_k^*(\psi_k)}{p_{k,\nu-1}^j(\psi_k)} \right] \right|. \quad (15)$$

Let us define the function $\alpha_{k,\nu}^j(\mathbf{x}_k)$ such that $\alpha_{k,\nu}^j(\mathbf{x}_k) = \left[\frac{p_{k,\nu}^j(\mathbf{x}_k)}{p_k^*(\mathbf{x}_k)} \frac{p_k^*(\psi_k)}{p_{k,\nu}^j(\psi_k)} \right]$ if $p_{k,\nu}^j(\mathbf{x}_k)p_k^*(\psi_k) \geq p_k^*(\mathbf{x}_k)p_{k,\nu}^j(\psi_k)$ and $\alpha_{k,\nu}^j(\mathbf{x}_k) = \left[\frac{p_k^*(\mathbf{x}_k)}{p_{k,\nu}^j(\mathbf{x}_k)} \frac{p_{k,\nu}^j(\psi_k)}{p_k^*(\psi_k)} \right]$ otherwise. Note that $\alpha_{k,\nu}^j(\mathbf{x}_k)$ is a continuous function since it is a product of continuous functions. Since $\alpha_{k,\nu}^j(\mathbf{x}_k) \geq 1$ and $\ln(\alpha_{k,\nu}^j(\mathbf{x}_k)) \geq 0$, $\forall \mathbf{x}_k \in \mathcal{X}$, (15) simplifies to:

$$\ln(\alpha_{k,\nu}^j(\mathbf{x}_k)) \leq \sigma_{m-1}(P_k) \ln(\alpha_{k,\nu-1}^j(\mathbf{x}_k)).$$

$$\alpha_{k,\nu}^j(\mathbf{x}_k) \leq (\alpha_{k,0}^j(\mathbf{x}_k))^{(\sigma_{m-1}(P_k))^\nu}. \quad (16)$$

Since $p_{k,\nu}^j(\mathbf{x}_k)$ tends to $p_k^*(\mathbf{x}_k)$, i.e., $\lim_{\nu \rightarrow \infty} \alpha_{k,\nu}^j(\mathbf{x}_k) = 1$, we can write (16) as:

$$\alpha_{k,\nu}^j(\mathbf{x}_k) - 1 \leq (\alpha_{k,0}^j(\mathbf{x}_k))^{(\sigma_{m-1}(P_k))^\nu} - 1^{(\sigma_{m-1}(P_k))^\nu}. \quad (17)$$

Using the mean value theorem (cf. [50]), the right hand side of (17) can be simplified to (18), for some $c \in [1, \alpha_{k,0}^j(\mathbf{x}_k)]$.

$$\begin{aligned} & (\alpha_{k,0}^j(\mathbf{x}_k))^{(\sigma_{m-1}(P_k))^\nu} - 1^{(\sigma_{m-1}(P_k))^\nu} \\ &= (\sigma_{m-1}(P_k))^\nu \left(c^{(\sigma_{m-1}(P_k))^\nu - 1} \right) (\alpha_{k,0}^j(\mathbf{x}_k) - 1). \end{aligned} \quad (18)$$

As $\sigma_{m-1}(P_k) < 1$, the maximum value of $(c^{(\sigma_{m-1}(P_k))^\nu - 1})$ is 1. Substituting this result into (17) gives:

$$\alpha_{k,\nu}^j(\mathbf{x}_k) - 1 \leq (\sigma_{m-1}(P_k))^\nu (\alpha_{k,0}^j(\mathbf{x}_k) - 1). \quad (19)$$

Hence $\alpha_{k,\nu}^j(\mathbf{x}_k)$ exponentially converges to 1 with a rate faster or equal to $\sigma_{m-1}(P_k)$. Irrespective of the orientation of $\alpha_{k,\nu}^j(\mathbf{x}_k)$ and $\alpha_{k,0}^j(\mathbf{x}_k)$, (19) can be written as (20) by multiplying with $\frac{1}{\alpha_{k,\nu}^j(\mathbf{x}_k)}$ or $\frac{1}{\alpha_{k,0}^j(\mathbf{x}_k)}$, and then with $p_k^*(\mathbf{x}_k)$.

$$\begin{aligned} & \left| \frac{p_k^*(\psi_k)}{p_{k,\nu}^j(\psi_k)} p_{k,\nu}^j(\mathbf{x}_k) - p_k^*(\mathbf{x}_k) \right| \\ & \leq (\sigma_{m-1}(P_k))^\nu \left| \frac{p_k^*(\psi_k)}{p_{k,0}^j(\psi_k)} p_{k,0}^j(\mathbf{x}_k) - p_k^*(\mathbf{x}_k) \right|. \end{aligned} \quad (20)$$

As shown in the proof of Theorem 4, we can choose $\tilde{\psi}_k \in \mathcal{X}$

such that $p_{k,0}^j(\tilde{\psi}_k) = p_k^*(\tilde{\psi}_k)$. Now we discuss two cases to reduce the left hand side of (20) to $|p_{k,\nu}^j(\mathbf{x}_k) - p_k^*(\mathbf{x}_k)|$.

$$\begin{aligned} & \left| \frac{p_k^*(\tilde{\psi}_k)}{p_{k,\nu}^j(\tilde{\psi}_k)} p_{k,\nu}^j(\mathbf{x}_k) - p_k^*(\mathbf{x}_k) \right| \\ &= \begin{cases} \left| p_{k,\nu}^j(\mathbf{x}_k) - p_k^*(\mathbf{x}_k) + \left(\frac{p_k^*(\tilde{\psi}_k)}{p_{k,\nu}^j(\tilde{\psi}_k)} - 1 \right) p_{k,\nu}^j(\mathbf{x}_k) \right| & \text{if } \frac{p_k^*(\tilde{\psi}_k)}{p_{k,\nu}^j(\tilde{\psi}_k)} \geq 1 \\ \left| p_k^*(\mathbf{x}_k) - p_{k,\nu}^j(\mathbf{x}_k) + \left(1 - \frac{p_k^*(\tilde{\psi}_k)}{p_{k,\nu}^j(\tilde{\psi}_k)} \right) p_{k,\nu}^j(\mathbf{x}_k) \right| & \text{if } \frac{p_k^*(\tilde{\psi}_k)}{p_{k,\nu}^j(\tilde{\psi}_k)} < 1 \end{cases} \\ & \geq |p_k^*(\mathbf{x}_k) - p_{k,\nu}^j(\mathbf{x}_k)|. \end{aligned}$$

Hence, for both the cases, we are able to simplify (20) to:

$$|p_{k,\nu}^j(\mathbf{x}_k) - p_k^*(\mathbf{x}_k)| \leq (\sigma_{m-1}(P_k))^\nu |p_{k,0}^j(\mathbf{x}_k) - p_k^*(\mathbf{x}_k)|.$$

Thus each $\mathcal{F}_{k,\nu}^j = p_{k,\nu}^j(\mathbf{x}_k)$ globally exponentially converges to $\mathcal{F}_k^* = p_k^*(\mathbf{x}_k)$ with a rate faster or equal to $\sigma_{m-1}(P_k)$. ■

The KL divergence is a measure of the information lost when the consensual pdf is used to approximate the locally estimated posterior pdfs. We now show that the consensual pdf \mathcal{F}_k^* obtained using Theorem 5, which is the weighted geometric average of the locally estimated posterior pdfs $\mathcal{F}_{k,0}^j, \forall j \in \{1, \dots, m\}$, minimizes the information lost during the consensus stage because it minimizes the sum of KL divergences with those pdfs.

Theorem 6. *The consensual pdf \mathcal{F}_k^* given by (14) globally minimizes the sum of Kullback–Leibler (KL) divergences with the locally estimated posterior pdfs at the start of the consensus stage $\mathcal{F}_{k,0}^j, \forall j \in \{1, \dots, m\}$, i.e.,*

$$\mathcal{F}_k^* = \arg \min_{\rho \in \mathcal{L}_1(\mathcal{X})} \sum_{i=1}^m D_{KL}(\rho || \mathcal{F}_{k,0}^i), \quad (21)$$

where $\mathcal{L}_1(\mathcal{X})$ is the set of all pdfs over the state space \mathcal{X} satisfying Assumption 7.

Proof: The sum of the KL divergences of a pdf $\rho \in \mathcal{L}_1(\mathcal{X})$ with the locally estimated posterior pdfs is given by:

$$\begin{aligned} & \sum_{i=1}^m D_{KL}(\rho || \mathcal{F}_{k,0}^i) = \\ & \sum_{i=1}^m \int_{\mathcal{X}} (\rho(\mathbf{x}_k) \ln(\rho(\mathbf{x}_k)) - \rho(\mathbf{x}_k) \ln(p_{k,0}^i(\mathbf{x}_k))) d\mu(\mathbf{x}_k). \end{aligned} \quad (22)$$

Under Assumption 7, $D_{KL}(\rho || \mathcal{F}_{k,0}^i)$ is well defined for all agents. Differentiating (22) with respect to ρ using Leibniz integral rule [54, Theorem A.5.1, pp. 372], and equating it to zero gives:

$$\sum_{i=1}^m \int_{\mathcal{X}} (\ln(\rho(\mathbf{x}_k)) + 1 - \ln(p_{k,0}^i(\mathbf{x}_k))) d\mu(\mathbf{x}_k) = 0, \quad (23)$$

where $\rho^*(\mathbf{x}_k) = \frac{1}{e} \prod_{i=1}^m (p_{k,0}^i(\mathbf{x}_k))^{1/m}$ is the solution to (23). The projection of ρ^* on the set $\mathcal{L}_1(\mathcal{X})$, obtained by

normalizing ρ^* to 1, is the consensual pdf $\mathcal{F}_k^* \in \mathcal{L}_1(\mathcal{X})$ given by (14).

The KL divergence is a convex function of pdf pairs [52, Theorem 2.7.2, pp. 30], hence the sum of KL divergences (22) is a convex function of ρ . If $\rho_1, \rho_2, \dots, \rho_n \in \mathcal{L}_1(\mathcal{X})$ and $\eta_1, \eta_2, \dots, \eta_n \in [0, 1]$ such that $\sum_{i=1}^n \eta_i = 1$, then $\rho^\dagger = \sum_{i=1}^n \eta_i \rho_i \in \mathcal{L}_1(\mathcal{X})$; because (i) since $\rho_i(\mathbf{x}_k) > 0, \forall \mathbf{x}_k \in \mathcal{X}, \forall i \in \{1, \dots, n\}$ therefore $\rho^\dagger(\mathbf{x}_k) > 0, \forall \mathbf{x}_k \in \mathcal{X}$; and (ii) since $\int_{\mathcal{X}} \rho_i(\mathbf{x}_k) d\mu(\mathbf{x}_k) = 1, \forall i \in \{1, \dots, n\}$ therefore $\int_{\mathcal{X}} \rho^\dagger(\mathbf{x}_k) d\mu(\mathbf{x}_k) = 1$. Moreover, since \mathcal{X} is a compact set, therefore $\mathcal{L}_1(\mathcal{X})$ is a closed set. Hence $\mathcal{L}_1(\mathcal{X})$ is a closed convex set. Hence (21) is a convex optimization problem.

The gradient of $\sum_{i=1}^m D_{KL}(\rho \| \mathcal{F}_{k,0}^i)$ evaluated at \mathcal{F}_k^* is a constant, i.e.,

$$\left. \frac{d}{d\rho} \sum_{i=1}^m D_{KL}(\rho \| \mathcal{F}_{k,0}^i) \right|_{\rho=\mathcal{F}_k^*} = m \ln \frac{e}{\int_{\mathcal{X}} \prod_{i=1}^m (p_{k,0}^i(\mathbf{x}_k))^{\frac{1}{m}} d\mu(\mathbf{x}_k)}.$$

This indicates that for further minimizing the convex cost function, we have to change the normalizing constant of \mathcal{F}_k^* , which will result in exiting the set $\mathcal{L}_1(\mathcal{X})$. Hence \mathcal{F}_k^* is the global minimum of the convex cost function (21) in the convex set $\mathcal{L}_1(\mathcal{X})$. This is illustrated using a simple example in Fig. 4.

Another proof approach involves taking the logarithm, in the KL divergence formula, to the base $c := \left(\int_{\mathcal{X}} \prod_{i=1}^m (p_{k,0}^i(\mathbf{x}_k))^{\frac{1}{m}} d\mu(\mathbf{x}_k) \right)$. Then differentiating $\sum_{i=1}^m D_{KL}(\rho \| \mathcal{F}_{k,0}^i)$ with respect to ρ gives:

$$\sum_{i=1}^m \int_{\mathcal{X}} (\log_c(\rho(\mathbf{x}_k)) + 1 - \log_c(p_{k,0}^i(\mathbf{x}_k))) d\mu(\mathbf{x}_k) = 0,$$

which is minimized by \mathcal{F}_k^* . Hence \mathcal{F}_k^* is indeed the global minimum of the convex optimization problem (21). ■

Note that if a central agent receives all the locally estimated posterior pdfs ($\mathcal{F}_{k,0}^j, \forall j \in \{1, \dots, m\}$) and is tasked to find the best estimate in the information theoretic sense, then it would also yield the same consensual pdf \mathcal{F}_k^* given by (14). Hence we claim to have achieved distributed estimation using this algorithm.

In Remark 5, we state that the methods for recursively combining probability distributions to reach a consensual distribution are limited to LinOP, LogOP, and their affine combinations.

Remark 5. The LinOP and LogOP methods for combining probability distributions can be generalized by the g-Quasi-Linear Opinion Pool (g-QLOP), which is described by the following equation:

$$\mathcal{F}_{k,\nu}^j = \frac{g^{-1}(\sum_{\ell \in \mathcal{J}_k^j} \alpha_{k,\nu-1}^{\ell} g(\mathcal{F}_{k,\nu-1}^{\ell}))}{\int_{\mathcal{X}} g^{-1}(\sum_{\ell \in \mathcal{J}_k^j} \alpha_{k,\nu-1}^{\ell} g(\mathcal{F}_{k,\nu-1}^{\ell})) d\mu(\mathbf{x}_k)}, \quad \forall j \in \{1, \dots, m\}, \forall \nu \in \mathbb{N}, \quad (24)$$

where g is a continuous, strictly monotone function. It is shown in [60] that, other than the linear combination of LinOP and LogOP, there is no function g for which the final consensus can be expressed by the following equation:

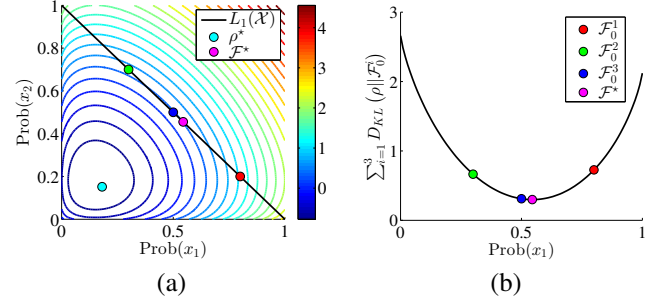


Fig. 4. Let the discrete state space \mathcal{X} have only two states x_1 and x_2 . All valid pmfs must lie on the set $\mathcal{L}_1(\mathcal{X})$ where $\mathbb{P}(x_1) + \mathbb{P}(x_2) = 1$. Given three initial pmfs $\mathcal{F}_0^i, i = \{1, 2, 3\}$, the objective is to find the pmf that globally minimizes the convex cost function $\sum_{i=1}^3 D_{KL}(\rho \| \mathcal{F}_0^i)$. In (a), $\rho^* = \frac{1}{e} \prod_{i=1}^3 (\mathcal{F}_0^i)^{1/3}$ globally minimizes the cost function, but it does not lie on $\mathcal{L}_1(\mathcal{X})$. In (b), $\mathcal{F}^* \in \mathcal{L}_1(\mathcal{X})$, which is the projection of ρ^* on the set $\mathcal{L}_1(\mathcal{X})$ obtained by normalizing ρ^* to 1, indeed globally minimizes the cost function on the set $\mathcal{L}_1(\mathcal{X})$.

$$\lim_{\nu \rightarrow \infty} \mathcal{F}_{k,\nu}^j = \mathcal{F}_k^* = \frac{g^{-1}(\sum_{j=1}^m \pi_j g(\mathcal{F}_{k,0}^j))}{\int_{\mathcal{X}} g^{-1}(\sum_{j=1}^m \pi_j g(\mathcal{F}_{k,0}^j)) d\mu(\mathbf{x}_k)}, \quad \forall j \in \{1, \dots, m\}, \forall \nu \in \mathbb{N}, \quad (25)$$

where π is the unique stationary solution. Moreover, the function g is said to be k-Markovian if the scheme for combining probability distribution (24) yields the consensus (25) for every regular communication network topology and for all initial positive densities. It is also shown that g is k-Markovian if and only if the g-QLOP is either LinOP or LogOP [60]. □

C. Communicating Probability Distributions

The consensus algorithms using either LinOP or LogOP need the estimated pdfs to be communicated to other agents in the network. We propose to adopt the following methods for communicating pdfs.

The first approach involves approximating the pdf by a weighted sum of Gaussians and then transmitting this approximate distribution. Let $\mathcal{N}(\mathbf{x}_k - m_i, B_i)$ denote the Gaussian density function, where the mean is the n_x -vector m_i and the covariance is the positive-definite symmetric matrix B_i . The Gaussian sum approximations lemma of [62, pp. 213] states that any pdf $\mathcal{F} = p(\mathbf{x}_k)$ can be approximated as closely as desired in the $\mathcal{L}_1(\mathbb{R}^{n_x})$ space by a pdf of the form $\hat{\mathcal{F}} = \hat{p}(\mathbf{x}_k) = \sum_{i=1}^{n_g} \alpha_i \mathcal{N}(\mathbf{x}_k - m_i, B_i)$, for some integer n_g and positive scalars α_i with $\sum_{i=1}^{n_g} \alpha_i = 1$. For an acceptable communication error $\varepsilon_{\text{comm}} > 0$, there exists n_g, α_i, m_i and B_i such that $\|\mathcal{F} - \hat{\mathcal{F}}\|_{\mathcal{L}_1} \leq \varepsilon_{\text{comm}}$. Several techniques for estimating the parameters are discussed in the Gaussian mixture model literature, like maximum likelihood (ML) and maximum a posteriori (MAP) parameter estimation [63]–[65]. Hence, in order to communicate the pdf $\hat{\mathcal{F}}$, the agent needs to transmit $\frac{1}{2} n_g n_x (n_x + 3)$ real numbers.

Let us study the effect of this communication error $\varepsilon_{\text{comm}}$ on the LinOP consensual distribution. Let $\tilde{\mathcal{F}}_{k,\nu}^j$ be the LinOP solution after combining local pdfs corrupted by communication error, i.e., $\tilde{\mathcal{F}}_{k,\nu}^j := \mathcal{T}(\cup_{\ell \in \mathcal{J}_k^j} \{\hat{\mathcal{F}}_{k,\nu-1}^{\ell}\})$ where $\mathcal{T}(\cdot)$ is

LinOP (6). We prove by induction that $\|\mathcal{F}_{k,\nu}^j - \tilde{\mathcal{F}}_{k,\nu}^j\|_{\mathcal{L}_1} \leq \nu \varepsilon_{\text{comm}}, \forall \nu \in \mathbb{N}$, where $\mathcal{F}_{k,\nu}^j$ is the true solution obtained from uncorrupted local pdfs. As the basis of induction holds, the inductive step for the ν^{th} consensus step is as follows:

$$\|\mathcal{F}_{k,\nu}^j - \tilde{\mathcal{F}}_{k,\nu}^j\|_{\mathcal{L}_1} \leq \sum_{\ell \in \mathcal{J}_k^j} a_{k,\nu-1}^{j\ell} \left(\|\mathcal{F}_{k,\nu-1}^\ell - \tilde{\mathcal{F}}_{k,\nu-1}^\ell\|_{\mathcal{L}_1} + \|\tilde{\mathcal{F}}_{k,\nu-1}^\ell - \tilde{\mathcal{F}}_{k,\nu-1}^j\|_{\mathcal{L}_1} \right) \leq (\nu-1)\varepsilon_{\text{comm}} + \varepsilon_{\text{comm}}. \quad (26)$$

Similarly, it follows from the proof of Theorem 5 that the LogOP solution after n_{loop} iterations ($\tilde{\mathcal{F}}_{k,n_{\text{loop}}}^j$), under communication inaccuracies, is always within a ball of $n_{\text{loop}}\varepsilon_{\text{comm}}$ radius from the true solution using LogOP ($\mathcal{F}_{k,n_{\text{loop}}}^j$) in the $\mathcal{L}_1(\mathcal{X})$ space, i.e., $\|\mathcal{F}_{k,n_{\text{loop}}}^j - \tilde{\mathcal{F}}_{k,n_{\text{loop}}}^j\|_{\mathcal{L}_1} \leq n_{\text{loop}}\varepsilon_{\text{comm}}$.

If particle filters are used to evaluate the Bayesian filter and combine the pdfs [36], [65], then the resampled particles represent the agent's estimated pdf of the target. Hence communicating pdfs is equivalent to transmitting these resampled particles.

The information theoretic approach for communicating pdfs is discussed in [66]. Let the local pdf $\mathcal{F}_{k,\nu}^j$ be transmitted over a communication channel using a finite sequence and the pdf $\tilde{\mathcal{F}}_{k,\nu}^j$ is reconstructed by the other agent. For a given error threshold, the minimum rate such that the variational distortion between $\mathcal{F}_{k,\nu}^j$ and $\tilde{\mathcal{F}}_{k,\nu}^j$ is bounded by the error threshold, is given by the mutual information between transmitted and received finite sequences.

Now that we have established that communication of pdfs is possible, let us discuss the complete BCF algorithm.

Algorithm 1 BCF–LogOP on SC Balanced Digraphs

- | | |
|--|-----------------|
| 1: (one cycle of j^{th} agent during k^{th} time instant) | |
| 2: Given the pdf from previous time step
$\mathcal{F}_{k-1}^j = p_{k-1}^j(\mathbf{x}_{k-1})$ | |
| 3: Set n_{loop} , the weights $a_{k,\nu}^{j\ell}$ | } Theorems 5, 7 |
| 4: while tracking do | |
| 5: Compute the prior pdf
$p_k^j(\mathbf{x}_k)$ using (3) | } Bayesian |
| 6: Compute the posterior pdf
$\mathcal{F}_k^j = p_k^j(\mathbf{x}_k \mathbf{z}_k^{S_k^j})$ using (4)
and measurement array $\mathbf{z}_k^{S_k^j}$ | Filtering |
| | Stage |
| | (Sec. II-B) |
| 7: for $\nu = 1$ to n_{loop} | |
| 8: if $\nu = 1$ then Set $\mathcal{F}_{k,0}^j = \mathcal{F}_k^j$
end if | |
| 9: Obtain the communicated
pdfs $\mathcal{F}_{k,\nu-1}^\ell, \forall \ell \in \mathcal{J}_k^j$ | } LogOP-based |
| 10: Compute the new pdf $\mathcal{F}_{k,\nu}^j$
using the LogOP (8) | Consensus |
| | Stage |
| | (Sec. III-B) |
| 11: end for
Set $\mathcal{F}_k^j = \mathcal{F}_{k,n_{\text{loop}}}^j$ | |
| end while | |
-

IV. MAIN ALGORITHMS: BAYESIAN CONSENSUS FILTERING

In this section, we finally solve the complete problem statement for BCF discussed in Section II-A and **Algorithm 1**.

We also introduce an hierarchical algorithm that can be used when some agents in the network fail to observe the target.

A. Bayesian Consensus Filtering

The BCF is performed in two steps: (i) each agent locally estimates the pdf of the target's states using a Bayesian filter with/without measurements from neighboring agents, as discussed in Section II-B, and (ii) during the consensus stage, each agent recursively transmits its pdf estimate of the target's states to other agents, receives estimates of its neighboring agents, and combines them using the LogOP as discussed in Section III-B. According to [67], this strategy of first updating the local estimate and then combining these local estimates to achieve a consensus is stable and gives the best performance in comparison with other update–combine strategies. In this section, we compute the number of consensus loops (n_{loop} in **Algorithm 1**) needed to reach a satisfactory consensus estimate across the network and discuss the convergence of this algorithm.

Definition 6. (*Disagreement vector $\theta_{k,\nu}$*) Let us define $\theta_{k,\nu} := (\theta_{k,\nu}^1, \dots, \theta_{k,\nu}^m)^T$, where $\theta_{k,\nu}^j := \|\mathcal{F}_{k,\nu}^j - \mathcal{F}_k^*\|_{\mathcal{L}_1}$. Since the \mathcal{L}_1 distance between pdfs is upper bounded by 2, the ℓ_2 norm of the disagreement vector ($\|\theta_{k,\nu}\|_{\ell_2}$) is upper bounded by $2\sqrt{m}$. \square

This conservative bound is used to obtain the minimum number of consensus loops for achieving ε -consensus across the network, while tracking a moving target. Let us now quantify the divergence of the local pdfs during the Bayesian filtering stage.

Definition 7. (*Error propagation dynamics $\Gamma(\cdot)$*) Let us assume that the dynamics of the ℓ_2 norm of the disagreement vector during the Bayesian filtering stage can be obtained from the target dynamics and measurement models (1) and (2). The error propagation dynamics $\Gamma(\cdot)$ estimates the maximum divergence of the local pdfs during the Bayesian filtering stage, i.e., $\|\theta_{k,0}\|_{\ell_2} \leq \Gamma(\|\theta_{k-1,n_{\text{loop}}}\|_{\ell_2})$, where $\|\theta_{k-1,n_{\text{loop}}}\|_{\ell_2}$ is the disagreement vector with respect to \mathcal{F}_{k-1}^* at the end of the consensus stage during the $(k-1)^{\text{th}}$ time instant; and $\|\theta_{k,0}\|_{\ell_2}$ is the disagreement vector with respect to \mathcal{F}_k^* after the update stage during the k^{th} time instant. \square

Next we obtain the minimum number of consensus loops for achieving ε -consensus across the network and also derive conditions on the communication network topology for a given number of consensus loops.

Theorem 7. (BCF–LogOP on SC Balanced Digraphs) *Under Assumptions 5, 7, 8, and an acceptable communication error $\varepsilon_{\text{comm}} > 0$, each agent tracks the target using the BCF algorithm. For some acceptable consensus error $\varepsilon_{\text{cons}} > 0$ and $\gamma_k = \min(\Gamma(\|\theta_{k-1,n_{\text{loop}}}\|_{\ell_2}), 2\sqrt{m})$:*
(i) for a given P_k , if the number of consensus loops n_{loop} satisfies

$$(\sigma_{m-1}(P_k))^{n_{\text{loop}}} \gamma_k + 2n_{\text{loop}}\varepsilon_{\text{comm}}\sqrt{m} \leq \varepsilon_{\text{cons}}; \quad (27)$$

or (ii) for a given n_{loop} , if the communication network topology (P_k) during the k^{th} time instant is such that

$$\sigma_{m-1}(P_k) \leq \left(\frac{\varepsilon_{\text{cons}} - 2n_{\text{loop}}\varepsilon_{\text{comm}}\sqrt{m}}{\gamma_k} \right)^{\frac{1}{n_{\text{loop}}}}; \quad (28)$$

then the ℓ_2 norm of the disagreement vector at the end of the consensus stage is less than $\varepsilon_{\text{cons}}$, i.e., $\|\theta_{k,n_{\text{loop}}}\|_{\ell_2} \leq \varepsilon_{\text{cons}}$.

Proof: In the absence of communication inaccuracies, Theorem 5 states that the local estimated pdfs $\mathcal{F}_{k,0}^j$ globally exponentially converges pointwise to a consensual pdf \mathcal{F}_k^* given by (14) with a rate of $\sigma_{m-1}(P_k)$, i.e. $\|\mathcal{F}_{k,\nu}^j - \mathcal{F}_k^*\|_{\mathcal{L}_1} \leq (\sigma_{m-1}(P_k))^\nu \|\mathcal{F}_{k,0}^j - \mathcal{F}_k^*\|_{\mathcal{L}_1}$. If $\theta_{k,0}$ is the initial disagreement vector at the start of the consensus stage, then $\|\theta_{k,n_{\text{loop}}}\|_{\ell_2} \leq (\sigma_{m-1}(P_k))^{n_{\text{loop}}} \|\theta_{k,0}\|_{\ell_2} \leq (\sigma_{m-1}(P_k))^{n_{\text{loop}}} \gamma_k$.

In the presence of communication error, combining (26) with the previous result gives $\|\tilde{\mathcal{F}}_{k,\nu}^j - \mathcal{F}_k^*\|_{\mathcal{L}_1} \leq (\sigma_{m-1}(P_k))^\nu \|\mathcal{F}_{k,0}^j - \mathcal{F}_k^*\|_{\mathcal{L}_1} + \nu\varepsilon_{\text{comm}}$. Since $\theta_{k,\nu}^j \leq \|\tilde{\mathcal{F}}_{k,\nu}^j - \mathcal{F}_k^*\|_{\mathcal{L}_1} + \nu\varepsilon_{\text{comm}}$, the disagreement vector after n_{loop} iterations is given by $\|\theta_{k,n_{\text{loop}}}\|_{\ell_2} \leq (\sigma_{m-1}(P_k))^{n_{\text{loop}}} \|\theta_{k,0}\|_{\ell_2} + 2n_{\text{loop}}\varepsilon_{\text{comm}}\sqrt{m}$. Thus, we get the conditions on n_{loop} or $\sigma_{m-1}(P_k)$ from the inequality $(\sigma_{m-1}(P_k))^{n_{\text{loop}}} \gamma_k + 2n_{\text{loop}}\varepsilon_{\text{comm}}\sqrt{m} \leq \varepsilon_{\text{cons}}$. ■

Note that in the absence of communication inaccuracies, (27) simplifies to $n_{\text{loop}} \geq \left\lceil \frac{\ln(\varepsilon_{\text{cons}}/\gamma_k)}{\ln \sigma_{m-1}(P_k)} \right\rceil$ and (28) simplifies to $\sigma_{m-1}(P_k) \leq \left(\frac{\varepsilon_{\text{cons}}}{\gamma_k} \right)^{\frac{1}{n_{\text{loop}}}}$. In the particular case where $n_{\text{loop}} = 1$ and communication errors are present, (28) simplifies to $\sigma_{m-1}(P_k) \leq \frac{\varepsilon_{\text{cons}} - 2\varepsilon_{\text{comm}}\sqrt{m}}{\gamma_k}$ and the necessary condition for a valid solution is $2\varepsilon_{\text{comm}}\sqrt{m} < \varepsilon_{\text{cons}}$. In the general case, it is desirable that $2\varepsilon_{\text{comm}}\sqrt{m} \ll \varepsilon_{\text{cons}}$ for a valid solution to Theorem 7.

B. Hierarchical Bayesian Consensus Filtering

In this section, we modify the original problem statement such that only m_1 out of m agents are able to observe the target at the k^{th} time instant. In this scenario, the other $m_2 (= m - m_1)$ agents are not able to observe the target. Without loss of generality, we assume that the first m_1 agents, i.e., $\{1, 2, \dots, m_1\}$, are tracking the target. During the Bayesian filtering stage, each *tracking agent* (i.e., agent tracking the target) estimates the posterior pdf of the target's states at the k^{th} time instant ($\mathcal{F}_k^j = p_k^j(\mathbf{x}_k | \mathbf{z}_k^{S_k^j \cap \{1, \dots, m_1\}})$, $\forall j \in \{1, \dots, m_1\}$) using the estimated prior pdf of the target's states (\mathcal{F}_{k-1}^j) and the new measurement array $\mathbf{z}_k^{S_k^j \cap \{1, \dots, m_1\}} := \{\mathbf{z}_k^\ell, \forall \ell \in S_k^j \cap \{1, \dots, m_1\}\}$ obtained from the neighboring tracking agents. Each *non-tracking agent* (i.e., agent not tracking the target) only propagates its prior pdf during this stage to obtain $p_k^j(\mathbf{x}_k)$, $\forall j \in \{m_1 + 1, \dots, m\}$.

The objective of hierarchical consensus algorithm is to guarantee pointwise convergence of each $\mathcal{F}_{k,\nu}^j$, $\forall j \in \{1, \dots, m\}$ to a pdf \mathcal{F}_k^* and only the local estimates of the agents tracking the target contribute to the consensual pdf. This is achieved by each tracking agent recursively transmitting its estimate of the target's states to other agents, only receiving estimates from its neighboring tracking agents and updating its estimate of the target. On the other hand, each non-tracking agent recursively

transmits its estimate of the target's states to other agents, receives estimates from all its neighboring agents and updates its estimate of the target. This is illustrated using the pseudo-code in **Algorithm 2** and the following equations:

$$\mathcal{F}_{k,\nu}^j = \mathcal{T} \left(\bigcup_{\ell \in \mathcal{J}_k^j \cap \{1, \dots, m_1\}} \{\mathcal{F}_{k,\nu-1}^\ell\} \right), \quad \forall j \in \{1, \dots, m_1\}, \forall \nu \in \mathbb{N}, \quad (29)$$

$$\mathcal{F}_{k,\nu}^j = \mathcal{T} \left(\bigcup_{\ell \in \mathcal{J}_k^j} \{\mathcal{F}_{k,\nu-1}^\ell\} \right), \quad \forall j \in \{m_1 + 1, \dots, m\}, \forall \nu \in \mathbb{N}, \quad (30)$$

where, $\mathcal{T}(\cdot)$ refers to the LogOP (8) for combining pdfs. Let \mathcal{D}_k represent the communication network topology of only the tracking agents.

Algorithm 2 Hierarchical BCF-LogOP on SC Balanced Digraphs

- | | | |
|--|---|--|
| 1: (one cycle of j^{th} agent during k^{th} time instant)
2: Given the pdf from previous time step
$\mathcal{F}_{k-1}^j = p_{k-1}^j(\mathbf{x}_{k-1})$
3: Set n_{loop} , the weights $a_k^{j\ell}$ } Theorems 7, 8
4: while tracking do
5: Compute prior pdf $p_k^j(\mathbf{x}_k)$ using (3)
6: if $j \leq m_1$ then
7: Compute the posterior pdf \mathcal{F}_k^j
using (4) and $\mathbf{z}_k^{S_k^j \cap \{1, \dots, m_1\}}$
8: end if
9: for $\nu = 1$ to n_{loop}
10: if $\nu = 1$ then
11: if $j \leq m_1$ then Set $\mathcal{F}_{k,0}^j = \mathcal{F}_k^j$
12: else Set $\mathcal{F}_{k,0}^j = p_k^j(\mathbf{x}_k)$ end if
13: end if
14: if $j \leq m_1$ then
15: Obtain the pdfs $\mathcal{F}_{k,\nu-1}^\ell$,
16: $\forall \ell \in \mathcal{J}_k^j \cap \{1, \dots, m_1\}$
17: from tracking neighbors
18: else Obtain the pdfs $\mathcal{F}_{k,\nu-1}^\ell$,
19: $\forall \ell \in \mathcal{J}_k^j$ from neighbors
20: end if
21: Compute the new pdf $\mathcal{F}_{k,\nu}^j$
22: using the LogOP (8)
23: end for
24: Set $\mathcal{F}_k^j = \mathcal{F}_{k,n_{\text{loop}}}^j$
25: end while | } | Bayesian
Filtering
Stage
(Sec. II-B) |
| 26: Obtain the pdfs $\mathcal{F}_{k,\nu-1}^\ell$,
27: $\forall \ell \in \mathcal{J}_k^j \cap \{1, \dots, m_1\}$
28: from tracking neighbors
29: else Obtain the pdfs $\mathcal{F}_{k,\nu-1}^\ell$,
30: $\forall \ell \in \mathcal{J}_k^j$ from neighbors
31: end if
32: Compute the new pdf $\mathcal{F}_{k,\nu}^j$
33: using the LogOP (8)
34: end for
35: Set $\mathcal{F}_k^j = \mathcal{F}_{k,n_{\text{loop}}}^j$
36: end while | } | Hierarchical
LogOP-based
Consensus
Stage
(Sec. IV-B) |
-

Assumption 9. The communication network topologies \mathcal{G}_k and \mathcal{D}_k are SC and the weights $a_k^{j\ell}$ are such that the digraph \mathcal{D}_k is balanced. The weights $a_k^{j\ell}$, $\forall j, \ell \in \{1, \dots, m\}$ and the matrix $P_{k,\nu-1}$ have the following properties: (i) The weights are the same for all consensus loops within each time instants, i.e., $a_k^{j\ell} = a_k^{\ell j}$ and $P_{k,\nu-1} = P_k$, $\forall \nu \in \mathbb{N}$. Moreover, P_k can be decomposed into four parts $P_k = \begin{bmatrix} P_{k1} & P_{k2} \\ P_{k3} & P_{k4} \end{bmatrix}$, where $P_{k1} \in \mathbb{R}^{m_1 \times m_1}$, $P_{k2} \in \mathbb{R}^{m_1 \times m_2}$, $P_{k3} \in \mathbb{R}^{m_2 \times m_1}$, and $P_{k4} \in \mathbb{R}^{m_2 \times m_2}$. (ii) If $j \in \{1, \dots, m_1\}$, then $a_k^{j\ell} > 0$ if and only if $\ell \in \mathcal{J}_k^j \cap \{1, \dots, m_1\}$, else $a_k^{j\ell} = 0$; hence $P_{k2} = \mathbf{0}^{m_1 \times m_2}$. Moreover, P_{k1} is balanced, i.e., $\sum_{\ell \in \mathcal{J}_k^j} a_k^{j\ell} = \sum_{r|j \in \mathcal{J}_k^r} a_k^{rj}$, where $j, \ell, r \in \{1, \dots, m_1\}$; (iii)

If $j \in \{m_1 + 1, \dots, m\}$, then $a_k^{j\ell} > 0$ if and only if $\ell \in \mathcal{J}_k^j$, else $a_k^{j\ell} = 0$; (iv) The matrix P_k is row stochastic, i.e., $\sum_{\ell=1}^m a_k^{j\ell} = 1$. \square

Theorem 8. (Hierarchical Consensus using the LogOP on SC Balanced Digraphs) *Under Assumptions 7 and 9, using the LogOP (8), each $\mathcal{F}_{k,\nu}^j$ globally exponentially converges pointwise to the pdf \mathcal{F}_k^* given by:*

$$\mathcal{F}_k^* = p_k^*(\mathbf{x}_k) = \frac{\prod_{i=1}^{m_1} \left(p_{k,0}^i(\mathbf{x}_k) \right)^{\frac{1}{m_1}}}{\int_{\mathcal{X}} \prod_{i=1}^{m_1} \left(p_{k,0}^i(\mathbf{x}_k) \right)^{\frac{1}{m_1}} d\mu(\mathbf{x}_k)} \quad (31)$$

at a rate faster or equal to $\sqrt{\lambda_{m_1-1}(P_{k1}^T P_{k1})} = \sigma_{m_1-1}(P_{k1})$. Only the initial estimates of the tracking agents contribute to the consensual pdf \mathcal{F}_k^* . Furthermore, their induced measures converge in total variation, i.e., $\lim_{\nu \rightarrow \infty} \mu_{\mathcal{F}_{k,\nu}^j} \xrightarrow{T.V.} \mu_{\mathcal{F}_k^*}, \forall j \in \{1, \dots, m\}$.

Proof: The matrix P_{k1} conforms to the balanced digraph \mathcal{D}_k . Let $\mathbf{1}_{m_1} = [1, 1, \dots, 1]^T$, with m_1 elements. Similar to the proof of Theorem 5, we get P_{k1} is a primitive matrix and $\lim_{\nu \rightarrow \infty} P_{k1}^\nu = \frac{1}{m_1} \mathbf{1}_{m_1} \mathbf{1}_{m_1}^T$.

Next, we decompose $\mathcal{U}_{k,\nu}$ from equation (11) into two parts such that $\mathcal{U}_{k,\nu} = [\mathcal{Y}_{k,\nu}; \mathcal{Z}_{k,\nu}]$, where $\mathcal{Y}_{k,\nu} = \left(\mathcal{H}_{k,\nu}^1, \dots, \mathcal{H}_{k,\nu}^{m_1} \right)^T$ and $\mathcal{Z}_{k,\nu} = \left(\mathcal{H}_{k,\nu}^{m_1+1}, \dots, \mathcal{H}_{k,\nu}^m \right)^T$. Since P_{k2} is a zero matrix, (11) generalizes and hierarchically decomposes to:

$$\mathcal{Y}_{k,\nu+1} = P_{k1}^\nu \mathcal{Y}_{k,0}, \forall \nu \in \mathbb{N} \quad (32)$$

$$\mathcal{Z}_{k,\nu+1} = P_{k3} \mathcal{Y}_{k,\nu} + P_{k4} \mathcal{Z}_{k,\nu}, \forall \nu \in \mathbb{N} \quad (33)$$

Combining equation (32) with the previous result gives $\lim_{\nu \rightarrow \infty} \mathcal{Y}_{k,\nu} = \frac{1}{m_1} \mathbf{1}_{m_1} \mathbf{1}_{m_1}^T \mathcal{Y}_{k,0}$. Thus $\lim_{\nu \rightarrow \infty} \mathcal{H}_{k,\nu}^j = \mathcal{H}_k^* = \frac{1}{m_1} \mathbf{1}_{m_1}^T \mathcal{Y}_{k,0} = \frac{1}{m_1} \sum_{i=1}^{m_1} \mathcal{H}_{k,0}^i, \forall j \in \{1, \dots, m_1\}$. From the proof of Theorem 5, we get $\mathcal{F}_{k,\nu}^j, \forall j \in \{1, \dots, m_1\}$ globally exponentially converges pointwise to \mathcal{F}_k^* given by (31) with a rate faster or equal to $\sigma_{m_1-1}(P_{k1})$.

Since $\mathcal{G}(k)$ is strongly connected, information from the tracking agents reach the non-tracking agents. Taking the limit of equation (33) and substituting the above result gives:

$$\lim_{\nu \rightarrow \infty} \mathcal{Z}_{k,\nu+1} = \frac{1}{m_1} P_{k3} \mathbf{1}_{m_1} \mathbf{1}_{m_1}^T \mathcal{Y}_{k,0} + P_{k4} \lim_{\nu \rightarrow \infty} \mathcal{Z}_{k,\nu} \quad (34)$$

Let $\mathbf{1}_{m_2} = [1, 1, \dots, 1]^T$, with m_2 elements. Since P_k is row stochastic, we get $P_{k3} \mathbf{1}_{m_1} = [\mathbf{I} - P_{k4}] \mathbf{1}_{m_2}$. Hence, from equation (34), we get $\lim_{\nu \rightarrow \infty} \mathcal{Z}_{k,\nu} = \frac{1}{m_1} \mathbf{1}_{m_2} \mathbf{1}_{m_1}^T \mathcal{Y}_{k,0}$. Moreover, the inessential states die out geometrically fast [68, pp. 120]. Hence $\lim_{\nu \rightarrow \infty} \mathcal{H}_{k,\nu}^j = \mathcal{H}_k^* = \frac{1}{m_1} \mathbf{1}_{m_1}^T \mathcal{Y}_{k,0}, \forall j \in \{m_1 + 1, \dots, m\}$. Hence, the estimates of the non-tracking agents $\mathcal{F}_{k,\nu}^j, \forall j \in \{m_1 + 1, \dots, m\}$ also converge pointwise geometrically fast to the same consensual pdf \mathcal{F}_k^* given by (31). By Lemma 2 we get $\lim_{\nu \rightarrow \infty} \mu_{\mathcal{F}_{k,\nu}^j} \xrightarrow{T.V.} \mu_{\mathcal{F}_k^*}, \forall j \in \{1, \dots, m\}$. \blacksquare

Note that Theorem 7 can be directly applied from Section IV-A to find the minimum number of consensus loops n_{loop} for achieving ϵ -convergence in a given communication network topology or for designing the P_{k1} matrix for a given number of consensus loops. A simulation example of Hierarchical

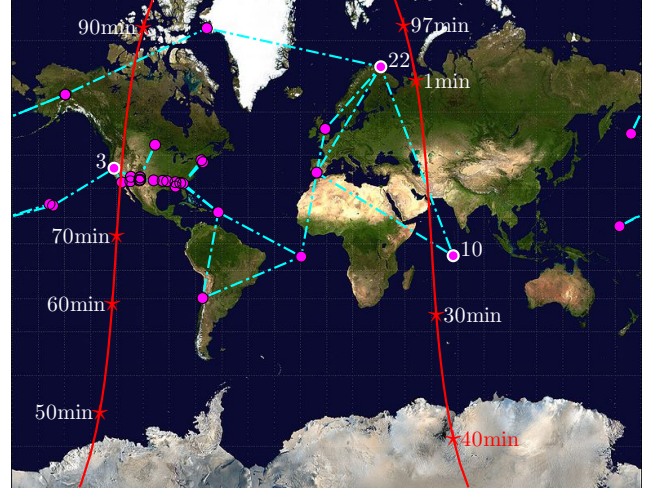


Fig. 5. The SSN locations are shown along with their static SC balanced communication network topology. The orbit of the Iridium-33 debris is shown in red, where \star marks its actual position during particular time instants.

BCF-LogOP algorithm for tracking orbital debris in space is discussed in the next section.

V. NUMERICAL EXAMPLE

Currently, there are over ten thousand objects in Earth orbit, of size 0.5 cm or greater, and almost 95% of them are nonfunctional space debris. These debris pose a significant threat to functional spacecraft and satellites in orbit. The US has established the Space Surveillance Network (SSN) for ground based observations of the orbital debris using radars and optical telescopes [69], [70]. In February 2009, the Iridium-33 satellite collided with the Kosmos-2251 satellite and a large number of debris fragments were created. In this section, we use the Hierarchical BCF-LogOP Algorithm to track one of the Iridium-33 debris created in this collision. The orbit of this debris around Earth and the location of SSN sensors are shown in Fig. 5.

The actual two-line element set (TLE) of the Iridium-33 debris was accessed from North American Aerospace Defense Command (NORAD) on 4th Dec 2013. The nonlinear Simplified General Perturbations (SGP4) model, which uses an extensive gravitational model and accounts for the drag effect on mean motion [71], [72], is used as the target dynamics model. The communication network topology of the SSN is assumed to be a static SC balanced graph, as shown in Figure 5. If the debris is visible above the sensor's horizon, then it is assumed to create a single measurement during each time step of one minute. The heterogeneous measurement model of the j^{th} sensor is given by:

$$\mathbf{z}_k^j = \mathbf{x}_k + \mathbf{w}_k^j, \text{ where } \mathbf{w}_k^j = \mathcal{N}(0, (1000 + 50j) \times \mathbf{I}),$$

where $\mathbf{x}_k \in \mathbb{R}^3$ is the actual location of the debris and the additive Gaussian measurement noise depends on the sensor number. Since it is not possible to implement the SGP4 target dynamics on distributed estimation algorithms discussed in the literature [11]–[20], we compare the performance of our Hierarchical BCF-LogOP algorithm (**Algorithm 2**) against the

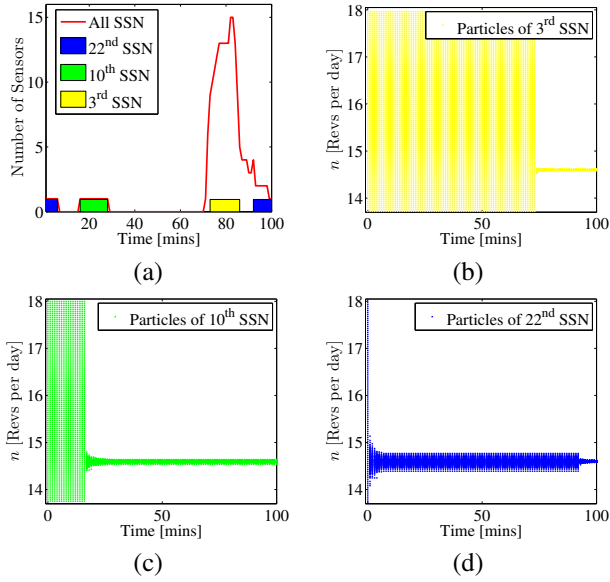


Fig. 6. (a) Number of SSN sensors observing debris. Trajectories of particles for stand-alone Bayesian filters for (b) 3rd, (c) 10th, and (d) 22nd SSN sensor.

Hierarchical BCF–LinOP algorithm, where the LinOP is used during the consensus stage.

In this simulation example, we simplify the debris tracking problem by assuming only the mean motion (n) of the debris is unknown. The objective of this simulation example is to estimate n of the Iridium–33 debris within 100 minutes. Hence, each sensor knows the other TLE parameters of the debris and an uniform prior distribution (\mathcal{F}_0^j) is assumed. Note that at any time instant, only a few of the SSN sensors can observe the debris, as shown in Fig 6(a). The results of three stand-alone Bayesian filters, implemented using particle filters with resampling [36], are shown in Fig 6(b-d). Note that the estimates of the 22nd and 10th sensors initially do not converge due to large measurement error, in spite of observing the debris for some time. The estimates of the 3rd sensor does converge when it is able to observe the debris after 70 minutes. Hence we propose to use the Hierarchical BCF–LogOP algorithm where the consensual distribution is updated as and when sensors observe the debris.

Particle filters with resampling are used to evaluate the Bayesian filters and communicate pdfs in the Hierarchical BCF algorithms. 100 particles are used by each sensor and 10 consensus loops are executed during each time step of one minute. The trajectories of all the particles of the sensors in the Hierarchical BCF algorithm using LinOP and LogOP and their respective consensual probability distributions at different time instants are shown in Figure 6(a-d). As expected, all the sensors converge on the correct value of n of 14.6 revs per day. The Hierarchical BCF–LinOP estimates are multimodal for the first 90 minutes. On the other hand, the Hierarchical BCF–LogOP estimates converges to the correct value within the first 10 minutes because the LogOP algorithm efficiently communicates the best consensual estimate to other sensors during each time step and achieves consensus across the network.

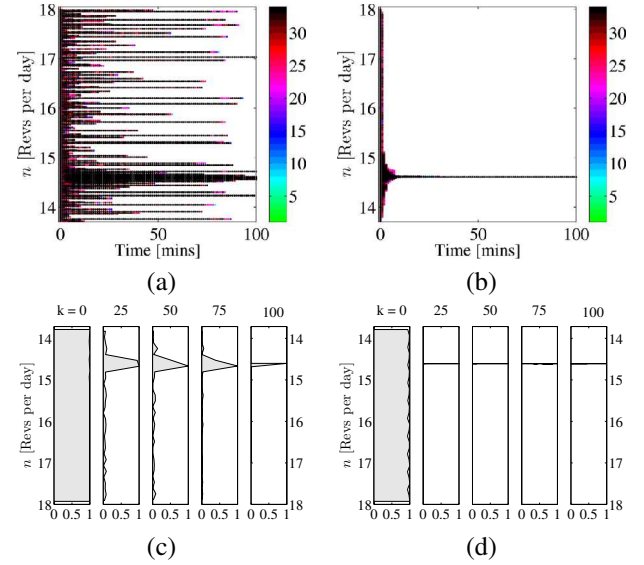


Fig. 7. Trajectories of particles of all sensors for (a) Hierarchical BCF–LinOP and (b) Hierarchical BCF–LogOP. The color-bar on the right denotes the 33 SSN sensors. Evolution of the consensual probability distribution for (c) Hierarchical BCF–LinOP and (d) Hierarchical BCF–LogOP.

VI. CONCLUSION

In this paper, we extended the scope of distributed estimation algorithms in a Bayesian filtering framework in order to simultaneously track targets, with general nonlinear time-varying target dynamic models, using a strongly connected network of heterogeneous agents, with general nonlinear time-varying measurement models. We introduced the Bayesian filter with/without measurement exchange to generate local estimated pdfs of the target’s states. We compared the LinOP and LogOP methods of combining local posterior pdfs and determined that LogOP is the superior scheme. The LogOP algorithm on SC balanced digraph converges globally exponentially, and the consensual pdf minimizes the information lost during the consensus stage because it minimizes the sum of KL divergences to each locally estimated probability distribution. We also explored several methods of communicating pdfs across the sensor network. We introduced the BCF algorithm, where the local estimated posterior pdfs of the target’s states are first updated using the Bayesian filter and then recursively combined during the consensus stage using LogOP, so that the agents can track a moving target and also maintain consensus across the network. Conditions for exponential convergence of the BCF algorithm and constraints on the communication network topology have been studied. The Hierarchical BCF algorithm, where some of the agents do not observe the target, has also been investigated. Simulation results demonstrate the effectiveness of the BCF algorithms for nonlinear distributed estimation problems.

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APPENDIX A PROOF OF THEOREM 3

Under Assumption 6, P_k is a nonnegative, row stochastic and irreducible matrix. Similar to the proof in [6], all the diagonal entries of P_k are positive, then $P_k^{m-1} > 0$ and [73, Theorem 8.5.2, pp. 516] implies that P_k is a primitive matrix. Since P_k is a regular matrix, it has only one recurrent class which is aperiodic [68, pp. 127]. Since P_k is row stochastic, $\mathbf{1}$ is its right eigenvector corresponding to the eigenvalue 1, i.e., $P_k \mathbf{1} = \mathbf{1}$. Moreover, according to the Gershgorin Disc Theorem (cf. [73, pp. 344]), all the eigenvalues of P_k are located in the unit circle, i.e. the spectral radius $\rho(P_k) = 1$. Hence Perron–Frobenius theorem (cf. [68, pp. 3]) states that there exists a left eigenvector π of P_k corresponding to the eigenvalue 1 which is unique to constant multiples, i.e., $P_k^T \pi = \pi$. The ergodic theorem for primitive Markov chains (cf. [68, pp. 119]) states that P_k has an unique stationary distribution given by the solution of the normalizing condition $\pi^T \mathbf{1} = 1$ and $\lim_{\nu \rightarrow \infty} P_k^\nu = \pi \pi^T$.

Obviously, (7) generalizes to $\mathcal{W}_{k,\nu} = P_k^\nu \mathcal{W}_{k,0}, \forall \nu \in \mathbb{N}$, where $\mathcal{W}_{k,0} = \left(\mathcal{F}_{k,0}^1, \dots, \mathcal{F}_{k,0}^m \right)^T$. Hence we get $\lim_{\nu \rightarrow \infty} \mathcal{W}_{k,\nu} = \pi \pi^T \mathcal{W}_{k,0}$. Thus, each $\mathcal{F}_{k,\nu}^j$ converges pointwise to the consensual pdf $\mathcal{F}_k^* = \pi^T \mathcal{W}_{k,0}$. Note that $\sum_{i=1}^m \pi_i = 1$, as needed for \mathcal{F}_k^* to be a valid probability distribution, where π_i is the individual element of the vector π . By Lemma 2, the measure induced by $\mathcal{F}_{k,\nu}^j$ on \mathcal{X} converges in total variation to the measure induced by \mathcal{F}_k^* on \mathcal{X} , i.e., $\lim_{\nu \rightarrow \infty} \mu_{\mathcal{F}_{k,\nu}^j} \xrightarrow{\text{T.V.}} \mu_{\mathcal{F}_k^*}$. ■

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